CORRECTIVE ARITHMETIC

For Supervisors, Teachers, and Teacher-Training Classes

BY

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UNDER THE EDITORSHIP OF

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EDITOR'S INTRODUCTION

People engaged in educational research have known for some time that a very distinctive piece of work was being done by the Director of Educational Measurements of the Wisconsin State Department of Public Instruction. This work was begun by Dr. W. W. Theisen, and it has been carried forward with unusual effectiveness by Dr. W. J. Osburn, the author of this book.

Dr. Osburn's contribution to the technique of teaching has consisted in the development of an analysis of errors and in the adjustment of remedial measures to the conditions exhibited by these errors. In this procedure he has used many of the existing means of measurement and has devised others. He has been peculiarly successful in securing the coöperation of the school people of Wisconsin, and his success in this direction is undoubtedly attributable to the recognized merit of his method. The various bulletins which he has issued to the schools of his State have attracted favorable attention in other States. These bulletins, however, were not issued in sufficient quantities to be available to teachers outside of Wisconsin nor, indeed, to any but a very few principals and superintendents. When, therefore, Dr. Osburn found himself in a position to put these matters, together with the rational basis upon which they rest, into the form of a book, it immediately became evident to those whom he consulted that such a systematic presentation of his doctrine would be of unquestionable advantage.

Although Dr. Osburn's interest was by no means confined to a single school subject, it is undoubtedly true that

his technique was more fully exhibited in the case of arithmetic than in the case of any other subject He was therefore right in formulating his method with reference to the subject on which he could furnish the greatest detail The result is the book which he now submits It is quite evidently a book which has grown out of actual practice, and as such it will make its greatest appeal to those who are engaged in actual teaching service It is essentially a book for teachers. In spite of that fact — indeed, I think because of it — it is of first-rate importance for those who train and supervise teachers.

In my judgment, those who read it will find in it a stimulating point of view especially as relates to the analysis of the errors which children make in arithmetic Customarily, we close our eyes to the nature and causes of error. Our attitude is reflected in the ratings which we give to pupils We say, for example, that a child's paper is worth seventy-five per cent, and by this we intend to measure the degree of his success Similarly, we record ratings in our classbooks and send them home to parents on report cards These ratings are our estimates of the pupils' success Instead of directing our attention and the attention of pupils and parents to the seventy-five per cent of success, we should undoubtedly accomplish more by considering the correlative twenty-five per cent of failure If we examine the errors which led to it, and if we try to locate the causes of these errors, we shall without question be taking the most effective way of increasing the seventy-five per cent of success.

When Dr Osburn looks into this matter of errors in arithmetic, he finds that they are typical and not merely of a haphazard character. He finds that the same sort of errors are committed in one city as in another and by children of every level of intelligence and home influence.

The fact that this is true makes it possible to devise a successful method of meeting these typical difficulties The way in which this is done, the types of material needed for the purposes, the penetrating analysis of the task of teaching arithmetic in the lower grades, the invoking of psychological principles to rationalize the procedure, the effective treatment of diagnosis, drill, and retesting — all these, woven into a new and stimulating method of teaching, have been presented concretely by the author of this book It is rightly called "Corrective Arithmetic." It proceeds, first, to set up what must be taught, second, to find out in what particulars pupils fall short of knowing the things they should know, third, to analyze the causes of their failure, fourth, to apply remedial measures, and, fifth, to furnish the material needed in carrying out the remedial measures.

In some ways Dr Osburn's book is strikingly original For example, he shows that not forty-five combinations in each of the fundamental processes, but a total of nearly seventeen hundred combinations must be taught Because a child can subtract 5 from 13 does not at all mean that he can subtract 5 from 43. Because he can multiply 6 by 7 does not mean that he can multiply 7 by 6. Dr Osburn asserts that we have trusted too implicitly to the doctrine of transfer of training "Is it any wonder," he says, "that children make mistakes? .. Let's give the child a chance"

The book is offered in the spirit of giving the child a chance It has limitations of some importance, and the author is himself clearly aware of them. Undoubtedly these limitations will evoke criticism, as indeed they should But to have waited until these limitations could have been eliminated would have been to delay the issuance of a book which despite its limitations appears to have pronounced merit. Those who counsel perfection would do

well to look about them and to ascertain who has a perfect plan in education and whether the fortunate possessor of such a plan is accomplishing as much as Dr Osburn is accomplishing with his plan in the schools of Wisconsin.

B. R. BUCKINGHAM

PREFACE

This book is largely a description of some research work which the author has been called upon to do in response to an insistent demand from teachers, supervisors, and superintendents for some means whereby better results might be obtained in the teaching of arithmetic. The problem was approached by means of a study of errors made by pupils in taking standardized tests. The results of this study have been presented in Chapters IV and V and to some extent in Chapter VII. The results of previous error studies were substantiated and it was proved definitely that errors are specific, widespread, and constant. The multiplicity of the necessary remedial work led to a review and restatement of well-known principles of economy in teaching as set forth in Chapter II. A further requirement essential in successful remedial work is some means by which the classroom teacher can diagnose the needs and disabilities of her pupils. This technique is presented in Chapter V.

After the teacher has become aware of the specific difficulties which her pupils are experiencing, suitable practice material must be available for the correction of the disabilities A search for such material soon disclosed the inadequate and poorly graded condition of the textbooks which are being used. This problem is discussed in Chapter VI.

Busy teachers often have to concentrate their attention on the more important things and neglect those which are less important The relative value of subject-matter is therefore discussed in Chapter III. Chapter VIII is a

condensed summary of the facts and laws of Educational Psychology as they apply to the teaching of arithmetic. Chapter VII gives specific suggestions for the work in each of the primary grades Chapter IX relates to means of motivation

A vast amount of research work remains to be done before the final word can be written concerning the teaching of arithmetic. In all probability the final word never can be written, but that is no reason why each worker in the field should cease striving toward the goal. In such a spirit this book has been written. The material which it contains is by no means complete. It aims only to suggest how to do a few things better than they have been done Practically all of the suggestions are now being profitably used in many of the schools of Wisconsin It is hoped that the book will prove interesting and helpful in other States also

The writing of a large part of the book would have been utterly impossible had it not been for the hearty and faithful cooperation of scores of superintendents, principals, supervisors, and teachers in Wisconsin whose names cannot be mentioned here. The author is deeply aware of the obligation which he is under to these people and is profoundly thankful to them for their cordial support

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CORRECTIVE ARITHMETIC

CHAPTER I

INTRODUCTION

CAN the results of our teaching in the fundamentals of arithmetic be improved? This is an old, old question, and it may seem to some that there is little need for another book in this field Scores of books have been written and teaching devices have greatly multiplied; yet the results of our efforts as teachers remain unsatisfactory. Courtis reports that only thirty per cent of our pupils attain a per cent of accuracy above seventy.[1] Business men everywhere deplore the lack of ability of their employees to perform with accuracy even the simplest of arithmetical processes Courtis's results show that their complaint is a legitimate one.

Consider, for example, the young applicant for a job, fresh from high school as he goes in search of work His prospective employer may well be expected to ask, "Can you add?" The answer, if truthful under present conditions, will of necessity be, "Well, I have completed all that the schools had to offer in arithmetic. I learned to add. Of course, I can get the correct answer to only six out of ten problems, but I received passing grades from my teachers" Yet the people are paying a substantial price to have their children made masters of these

[1] Brown, J C, and Coffman, L D *How to Teach Arithmetic*, p. 38. Chicago: Row, Peterson & Co, 1914

skills which are fundamental to successful living. If inquiries be started in search for the causes of this intolerable condition of affairs, the result is generally nothing further than a more or less skillful "passing of the buck" The school board passes it to the superintendent and supervisors, while they in turn pass it on to the teacher. The teacher, knowing as she generally does that she has labored long and hard, is much perplexed to find a legitimate excuse It is true that many pupils do not attend school as regularly as they should, and it is also true that there is an astonishing tendency for the facts of arithmetic to go into one ear of a pupil and out at the other. A few of the pupils are utterly incapable of learning the arithmetical facts and should not even be allowed to attempt them.

To make matters worse, the elementary-school curriculum is no longer a simple combination of the three R's One subject after another has pushed its way into the school We now have spelling, geography, history, language, hygiene, civics, music, art, and ever more insistent demands for economics and sociology. Supervised study and project teaching are adding to the demands made on the already overburdened teacher Within the life of the present generation teaching has changed from a relatively easy occupation to a tremendously exacting and nerve-racking one

At first sight the outlook for better results is gloomy; but it is the firm belief of the author of this book that there is a way out That way lies along the line of more economical methods of teaching based upon a more definite conception of the actual needs The author has no intention to discredit in any manner the splendid enrichment of curriculum and methods that is now in progress. On the contrary, he strongly favors them.

The purpose of this book is to help to make them possible by showing how to secure better results in the fundamentals in less time The studies upon which this work is based are still far from complete, but sufficient data are now available to warrant some suggestions concerning improvement which will be both helpful and constructive so far as the work of the first four grades is concerned.

CHAPTER II

PRINCIPLES OF ECONOMY IN TEACHING

The preceding chapter has set forth the problem which every primary teacher faces in her arithmetic teaching, and has suggested that a solution of the problem lies in the direction of more economical methods of teaching. The purpose of this chapter is to summarize a few of the more important principles to guide us in our efforts to save time and energy and at the same time to improve results It may seem at first sight that the principles which are to follow are too commonplace to be mentioned in a book which expects to make a contribution to our technique of teaching Such is not the case, however, because the actual evidence shows conclusively that quite the contrary is true This evidence will be presented from time to time throughout the book.

I Suit the Amount of Drill to the Difficulty of the Task

This is one of our oldest pedagogical precepts. Teachers and supervisors for generations past have granted its truth and have tried to apply it in their practice. The handicap which has always beset them is the fact that the actual amounts of difficulty have not been known in detail. It has been assumed that the particular number facts are all equally difficult for the child. Not until 1915, when Holloway [1] published his study of the relative difficulty of the fundamental combinations, was there

[1] Holloway, H. V *Experimental Study to Determine the Relative Difficulty of Elementary Number Combinations in Addition and Multiplication.* Philadelphia: University of Pennsylvania, 1917, 102 pp. (University of Pennsylvania Studies in Education.)

any adequate realization of the great difference in difficulty which exists between learning that 2 and 2 are 4 and learning that 9 and 7 are 16. Further details concerning these differences are given in Chapter VI. Here we shall have to content ourselves with saying that our present practice material is almost the exact reverse of what it should be so far as Principle I is concerned[1] Supplementary practice material to remedy this situation is indispensable if we hope to improve our results as they should and can be improved.

II Teach First and Most Completely That Which is Most Used in Life

This is another principle, the truth of which is self-evident; yet apparently not until 1919, when Wilson published his survey,[2] was any attention given to finding out what arithmetic is actually needed outside of the schoolroom. An effort has since been made to eliminate from our textbooks much of the material which is relatively useless socially, but the process is yet far from complete. It is still true that the people are paying taxes to support schools in which essential facts are poorly taught and in which non-essentials are taught reasonably well. Surely busy teachers can ill afford to continue such a policy In Chapter III this matter is considered in detail together with suggestions for improvement.

III Teach What the Pupils do not Know. Do not Teach What They Already Know

The author remembers as a boy that spelling lessons were assigned strictly according to the divisions set forth

[1] See Chapter III, pp 10 ff

[2] Wilson, G M *A Survey of the Social and Business Usage of Arithmetic* New York Teachers College, Columbia University, 1919, 62 pp (Teachers College, Columbia University Contributions to Education, no. 100.)

in the spelling book. The teacher never thought of making an effort to find out whether or not some in the class already knew how to spell every word in the lesson, yet such was often the case. Similar procedure is still to be found in most of our schools both in spelling and in arithmetic. Pupils are continually being required to study that which they already know On the other hand, pupils are failing to get the answers to their problems even in the high school, because they fail to respond correctly to a few fundamental number facts. The author recently found a boy in the demonstration school of a State normal school who was failing to get his answers correctly because he said that 8 times 7 are 63 every time he met it in his work How much discouragement and how much lack of self-confidence on the part of the pupil would be removed if we could teach only what the child needs instead of wasting our time and his in drilling him upon what he already knows This principle will be considered more in detail in Chapter IV.

IV. Do not Try to Teach What the Child Could not Learn even with the Best Teacher in the World

A large number of teachers are afflicted with one or two pupils who simply cannot learn. Strenuous efforts on the part of the teacher may bring forth the correct response to a new word or number fact for to-day, but when to-morrow comes the child has forgotten everything which he had seemingly learned the day before. Such children appeal to our sympathies and some of them have even learned how to act so as to heighten this appeal Thus it happens that some of our best teachers spend most of their precious nerve energy on the pupil who cannot profit by it. They are trying to teach something which the child cannot remember even if he were taught

by the best teacher whom the sun ever shone upon — while the rest of the children in the room are being robbed of their share of the teacher's time and attention. Many of these latter children are having their troubles also. Some of them are trying to divide when they cannot subtract, or multiply when they cannot add A little judicious help from the teacher would do them a great amount of permanent good The fundamental contention here is that the teacher should distribute her attention in accordance with *two* guiding principles instead of *one*. She must try to help all who need help, but in addition to this she must *attend first* to those who need help and *who can profit by it* To overlook the latter principle is to make a serious mistake, because unless this precept is observed, it will inevitably follow that the most neglected pupil in the school will be the most brilliant one. This question will be considered further in Chapter V.

V. ALWAYS DO YOUR BEST TO MAKE PUPILS WANT TO LEARN WHAT THEY SHOULD LEARN

Do not try to teach pupils when they are tired, bored, or sleepy A full discussion of this principle will be given in Chapters VIII and IX. Here it will be possible to indicate in a brief manner only some of the annoyances and distractions which the teacher must remove from her pupils if she is to avoid wasting her strength. It must be confessed, however, at the outset that entirely too little is known concerning the nature and operation of the things which annoy children and distract their attention from their work. On the other hand, some well-known facts are often overlooked. Most teachers know that children must be comfortable and contented if they are to learn. Surely every teacher has heard somewhere that the temperature must be suitable in all parts of the

room; that the seats must be comfortable, that unnecessary confusion should be avoided, and that the health and physical condition of the children should be looked after All of these things should be considered, yet it is certain that some of them are ignored in most of our schools.

The author recently visited a school where facts such as those mentioned above had been singularly disregarded As the door of the room was opened, the visitor was met with an outpouring of hot, foul air reeking with sickening odors The thermometer stood at eighty degrees Upon investigation it was found that the fresh-air intake had been carefully closed. Some of the children were sitting in seats too high for them, with their feet dangling above the floor Others were enduring similar torture in seats which were too low There were no shades to shut out the glaring light No attention had been paid to the physical condition of the pupils For all that the teacher knew, many of the pupils may have been suffering from poor eyesight or hearing, malnutrition, decayed teeth, bad tonsils, or adenoids Indeed, the gaping mouths of two or three pupils made the last condition almost a foregone conclusion Some of the children must have had headache Their faces were flushed and red and an uncontrollable drowsiness pervaded the whole room. In spite of all this, the teacher seemed earnest and conscientious; and the school was situated in a rather enlightened community.

While the author does not wish to imply that conditions are as bad as this in many schools, he is firmly of the opinion that one or more elements of this situation are present in the vast majority of them Certainly no industrial concern could operate with profit under such violation of the laws of economy and common sense.

Schools are expensive and teaching is hard work at best. Surely better conditions are at hand, and it is to be most sincerely hoped that within the lifetime of those now living we as a nation shall be able to look back upon this age of waste and extravagance of effort and marvel at our own stupidity.

CHAPTER III

RELATIVE VALUE OF SUBJECT-MATTER

ANY one who takes time to sum up all that a child is supposed to know about the fundamentals of arithmetic at the end of his fourth year in school will be struck with the magnitude of his task and will obtain an illuminating notion as to why our results are as poor as they are.

COMBINATIONS TO BE LEARNED

Let us look at the task somewhat in detail There are 100 combinations each in addition, subtraction, and multiplication In order to do column addition accurately up to and including all cases where 9 is to be carried, the child must know more than 700 additional facts.[1] In order to master short division, the child must know 450 facts[2] including the 90 cases where there are no remainders and the 360 cases in which remainders occur.

In long division, if one thinks only of the examples in which there are two digits in the divisor and two in the quotient, one finds that there are 8100 such exercises. It should be borne in mind that all of this takes no account of the work often done in these grades in common fractions and denominate numbers nor of the work in reasoning problems which is always done

The following is a summary of what we expect of the child

[1] These are the cases in which a number less than 10 is added mentally to 10 or to a number greater than 10 These are called high-decade additions For samples see Appendix, pp 163 to 165

[2] A complete list is given in the Appendix, pp 161 and 169 ff.

Simple addition	100 combinations
Higher-decade addition	755 combinations
Simple subtraction	100 combinations
The subtraction involved in short division	175 combinations
Simple multiplication	100 combinations
Short division	450 combinations
Total	1680 combinations

A truly stupendous task for the young mind!

The Inadequacy of Transfer

The real magnitude of this undertaking has been largely obscured by a pedagogical doctrine which we have thoughtlessly followed for generations This doctrine is known as the "transfer of training" According to it, if a child is taught a combination in direct form, he will always know it in reverse form If he learns it in the simple form, he will always know it in every form For example, if a child is taught how much 8 and 2 are, it is assumed that he will always respond correctly to 2 and 8, 18 and 2, 28 and 2, 12 and 8, 22 and 8, and so on. In like manner a child who is taught that 8 from 13 equals 5 is expected to respond correctly to 8 from 23, 8 from 33, 5 from 13, 45 from 53, and so on. Lastly, a child who is taught how much 7 times 8 are is supposed to respond correctly to 8 times 7, 56 divided by 7, 62 divided by 8, and the like Thus we have been encouraged to believe that all our work is done when we have taught 180 facts — the 45 so-called "principal combinations" in addition, subtraction, multiplication, and division Just how this marvelous transfer is to take place completely from 180 combinations to the 1680 has never been explained. Yet teachers still go about their work with the simple faith that the expected will infallibly take place

On the other hand, we have long known that the scheme

did not work Parents and business men have complained for more than a century that the results of our teaching in arithmetic are inadequate Little improvement resulted, however, until our teachers, textbook writers, and educators began to break away from the time-honored "armchair philosophy" which they had so long followed Instead of theorizing about how the teaching should be done, we find that since 1900 they have been making an honest effort to analyze what actually happens under the old plan and to study means of improving it through the use of modern scientific method. In 1903, in his *Educational Psychology*, Thorndike launched his first attack against the general theory of transfer of training or formal discipline as it was then called, Bagley following with his *Educative Process* in 1905 The work of these men has since been verified by many other investigators The substance of their conclusions is as follows: (1) Transfer is real in many situations, but is never perfect (2) It can take place only when there is either identity of elements, identity of procedure, or identity of ideals

There is, indeed, a partial identity of elements in the additions 5 and 8 and 15 and 8 There is also partial identity of procedure and there may be identity of ideals, such as pride in the correct answer, pertinacity, and the like. But the identity is never complete, and from this fact it necessarily follows that the transfer is also incomplete. This means that the old methods of teaching the fundamentals of arithmetic were bound to be faulty — a conclusion which is abundantly supported by the unsatisfactory character of the results which we have been getting

Thus we find ourselves facing the task of teaching children to master 1680 number facts with methods which

have proved to be inadequate. In seeking help in such a situation there are two principles which can guide us They have been mentioned in Chapter II, but they are repeated here for emphasis: (1) *Teach first that which is most used or should be most used in life outside of school* (2) *Suit the amount of drill to the difficulty of the task.*

Now, what facts in the fundamentals of arithmetic are of most use in life outside of school? It seems incredible that so little attention has been given to this question The first contribution to it so far as the writer knows was made by Wilson less than ten years ago [1] He found that the amount of arithmetic which people use is astonishingly small. Indeed, so little is used that some writers believe this is not an adequate criterion of what should be taught. Some competent people think it would be better to teach all the people *might* use with profit in the transaction of their business Yet, even if we adopt the latter criterion, the amount of arithmetic fundamentals which should be thoroughly mastered is relatively small From this point of view the writer proposes the following number facts as far too important in life outside of school to be neglected or left to be acquired through the limited possibilities of transfer

Combinations in Addition

It is assumed that everybody will need the ability to add, subtract, multiply, and divide accurately, at least within certain limits As to addition, every person should be able to add any number less than ten to any other number of the same sort In other words, he should be able to add every possible combination of 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 taken two at a time. This includes these

[1] Wilson, G M *A Survey of the Social and Business Usage of Arithmetic.* New York: Teachers College, Columbia University, 1919

combinations in both direct and reverse form For example, the addition of 2 and 7 is to be taught just as carefully as 7 and 2. It is also necessary to teach carefully all combinations involving zero Children do not inherit the ability to add zero and 6 or 8 and zero. The failure to teach the zero combinations is responsible for a large share of the error in the fundamentals of arithmetic even in high-school classes. The total number of these addition facts is 100.

Column Addition

A perfect knowledge of these 100 addition facts will enable a pupil to make every possible combination of any one number less than 10 with any other number less than 10. This knowledge is extremely important, but obviously it does not include all of the information which is needed in addition Every one occasionally meets a situation in which he needs to add more than two numbers together This process is usually called "column addition." A need for it arises most frequently in connection with the purchase of a bill of goods when it is required to find the total amount spent Such a need is exemplified in the following situation:

$$\begin{array}{r} \$\ \ 38 \\ 11.75 \\ 9\ 20 \\ 13\ 18 \\ .57 \\ 3\ 96 \\ \underline{2.98} \end{array}$$

Pupils are usually taught to begin at the lower right-hand corner. Let us suppose they have learned the 100 facts mentioned above, among them the fact that 8 and 6

are 14 The next step, however, is new. The child must now *think* 14, and add a written 7 to the 14 which is in the mind Having obtained 21, the child must now *think* 21 and add 8 to it In like manner the process is continued until the top of the column is reached. The final sum for the right-hand column is 42 The child must then write the 2 in the units' place and add 4 to the 9 at the bottom of the second column from the right (Addition upward is assumed) Four and 9 are 13 is one of the 100 facts already learned, but the addition of the remainder of the tens' column requires the mastery of additional facts.

Height of Columns

The process is just the same for each succeeding column as one goes from right to left in column addition. The new facts needed in column addition are evidently the higher decade combinations. As already stated, there are more than 700 of these, and any of them may be needed if the columns are high enough to give sums up to 100 In other words, the number of higher-decade additions which a child will use depends upon the height of the columns which he is expected to add accurately How tall should these columns be? No answer has ever been agreed upon so far as the author is aware The result is that some teachers are asking their pupils to add columns which extend from the bottom of their tablets to the top, while other teachers are giving little attention to any column addition.

Small practical need for tall columns It is evident that some more definite standards should be set if the principles of economy of teaching are to receive consideration In attempting to arrive at a standard, two or three facts should be considered which tend to lessen the em-

phasis to be placed on the adding of extremely tall columns. In the first place, there is little practical need for the addition of tall columns outside of banks and similar places of business. Almost invariably, however, these institutions own adding machines and have no need for employees who can add such columns "by hand."

In the second place, even though such addition is necessary in a few small grocery stores, meat markets, and the like, it must be remembered that one always starts the column, however tall it may be, with the lower decades One must add through the tens, twenties, and thirties in order to reach the fifties, sixties, and seventies. It follows, then, that additions in the lower decades are used much more frequently

The attention span as related to height of columns In the third place, there is a limit to the number of figures which a person can add in a column without a pause This is because each person has what is known as an "attention span." There is a limit to the amount of time which any of us can spend thinking closely about any one thing There is also a limit to the number of things which we can see at a glance or remember when heard as a group It is probable that the average child cannot add more than six or eight figures without a pause.[1] If a child attempts to add more than this, his attention wanders for a moment from the task in spite of himself. It is true that he can bring himself back to the work after a short pause, but by this time he may have forgotten the partial sum which he was holding in his mind and he may have forgotten just how far he had gone in his process of adding up the column In either case an error is inevitable This means that the difficulty of correct

[1] Courtis, S A *Teachers' Manual for Practice Tests in Arithmetic*, pp 38, 39. Yonkers, World Book Company, 1920

addition increases enormously as the column increases in height. It would be better, therefore, to break the extremely long column into two or three, add each one separately, and then add the two or three separate sums

The most important combinations For all of these reasons it seems advisable to set a limit to the height of the columns which we assign for addition. The author suggests that we teach first, and as thoroughly as possible, the addition of columns in which no column adds up to more than 39. If longer columns are considered at all, they should either be broken in two or left for spare time, if there is any, in the upper grades To assign addition exercises which are nine figures high and three or four figures across is useless and even cruel

If we make up our minds, then, to limit our efforts to the teaching of that which is most useful in column addition, the task is very much shortened. Instead of having more than 700 combinations in high-decade addition we shall now have only 255 Thirty of these, however, are of the form 16 and zero, 29 and zero, and the like. These cover all the cases in column addition up to sums of 39 in which zeros may occur in the column Now zeros are not of importance in column addition and the child should be taught merely to neglect them. After these 30 are deducted, 225 of the total 255 remain to be taught.[1]

Since these 225 are already partly known, it seems certain that, with a small amount of attention closely centered upon the most useful and most difficult of them, we shall be able to solve the old and troublesome question of how to get accurate results in column addition.

ADDITION IN MULTIPLICATION

One further sort of addition is fundamental and essential to the successful transaction of the business of life.

[1] See Appendix, pp. 163 to 165.

This is the addition which occurs in multiplication. Suppose one needs to multiply 98 by 7, for example. The problem is usually written in the form $\begin{array}{r}98\\ \underline{7}\end{array}$. In order to obtain the right answer some addition is necessary. The child must think 7 times 9 are 63 and must remember that he has 5 to carry. Here we have a situation where it is necessary to add two numbers *both* of which are in the mind *only*. They are never written. Now, 63 and 5 is one of the higher-decade additions mentioned on page 11, but it is not included in the minimum given on page 17, since 63 is greater than 39. It follows, then, that some of the combinations in the upper decades are much more important than others. For example, 63 and 5 is of great practical value, but 67 and 5 is not. This is because 63 is one of the products in the multiplication table while 67 is not If one takes the trouble to list all possible additions which can ever occur in multiplication situations like the one given above, one finds that there are 175 of them. Eighty-eight of these occur also in the group needed for the addition columns whose sums are 39 or less. It follows, therefore, that only 87 combinations are needed in addition to the 225 mentioned on page 17. The 88 are needed both in multiplication and in the most useful column addition It is recommended, therefore, that these be reviewed again in connection with the 87 new ones. No teacher can expect high accuracy in carrying in multiplication unless she has carefully and conscientiously taught each one of these 175 addition combinations.[1]

Combinations in Subtraction

In subtraction two sorts of situations need careful attention. They are the 100 combinations as ordinarily

[1] See Appendix, pp 165 to 166 for list of these 175 combinations

given in the subtraction table and the 175 subtraction combinations which are needed in short division To neglect either of these groups is to insure poor results in accuracy The first group is usually taught carefully, but the second is universally neglected even in our *best* textbooks When children meet the situation $8\overline{)712}$, they must subtract 64 (held in mind) from 71 But no previous practice has been given in this sort of thing. The only clue that even the bright child has is his knowledge that 4 from 11 leaves 7 Now, the question is what does 64 from 71 leave — a question which is considerably different Is it any wonder that children make mistakes? To attempt to teach the mastery of short division under such conditions is rank stupidity. Let's give the child a chance. *Teach him the 175 combinations.*[1]

Combinations in Multiplication

In multiplication there are 100 combinations which must be thoroughly mastered As in addition, it is poor policy to teach only 45 combinations. It is equally as bad to fail to teach the zero combinations It is not at all unusual to find a high-school pupil failing to get the correct result in an algebra exercise because he responds "8" to the situation 8 times zero

Combinations in Division

In division the requirements are more extended. If the child is to be expected to do accurately all possible forms in short division, he must have the mastery of 450 facts He may be called upon to make any of the following divisions.[2]

[1] See Appendix, pp 167–169, for list of combinations

[2] Division by all numbers greater than 9 is considered as long division in this book, although after the first learning stage, divisors 10, 11, and 12 should be handled by the short-division process

One	into	any	number	from	zero	to	9	inclusive
Two	"	"	"	"	"	"	19	"
Three	"	"	"	"	"	"	29	"
Four	"	"	"	"	"	"	39	"
Five	"	"	"	"	"	"	49	"
Six	"	"	"	"	"	"	59	"
Seven	"	"	"	"	"	"	69	"
Eight	"	"	"	"	"	"	79	"
Nine	"	"	"	"	"	"	89	"

The total number of possible divisions with 1 as a divisor is 10, with 2 as a divisor, 20; with 3 as a divisor, 30; and so on up to 9, which occurs as a divisor 90 times. Thus the grand total for all possible facts in short division is 450 Ninety of these are the facts which are ordinarily taught. They are the total number of cases in short division in which there are no remainders It has been customary until very recently to expect the child to attain a mastery of the 360 cases in which remainders occur without help from the teacher. Every teacher knows that this is impossible particularly in the 45 cases in which a child is called upon to divide a number into something less than itself. The following is an illustration:

$$\begin{array}{r} 34 \\ 6\overline{)1824} \end{array}$$

The correct answer, of course, is 304. The child has erred in failing to think "6 into 2 goes zero times" Failure to teach needed facts is the cause of this most prevalent error.

In general, it may be said that satisfactory accuracy in short division is not to be expected until *all* of the pupils have had *at least some practice* on each of the possible 450 combinations.

The summary on page 11 showed a total of 1680 combinations, which a child *might* need. The following

summary shows the number which he is *almost sure* to need after he leaves school.

Addition		412
Simple	100	
Higher decade in column addition when the sum of no column exceeds 39	225	
Additional higher decade for use in multiplication	87	
Subtraction		275
Simple	100	
Additional subtraction needed in short division	175	
Multiplication (simple)		100
Division		450
Total		1237

No teacher in grades four to eight can consider her work well done until she has made an inventory of the amount of mastery of these combinations attained by each of her pupils [1]

We have seen how a consideration of use outside of school has reduced the number of facts to be taught from 1680 to 1237. The second principle, quoted on page 13, enables us to go one step farther It tells us to suit the amount of drill to the difficulty of the task If perchance any teacher becomes so busy that she cannot possibly teach all of the 1237 facts, she should teach the ones that are most difficult, hoping that the child, in some way or other, will learn the easy ones himself. Which ones of the 1237 facts are the most difficult?

The More Difficult Combinations

Addition and subtraction Information on this question is still far from complete A beginning has been

[1] For a complete list see Appendix, pp 155 ff

made, however, and one point is well established The exercises which involve bridging are generally more difficult than those which do not [1]

Bridging means either adding or subtracting from one decade into another one — usually an adjacent one. All the integers less than 10 constitute the first decade; all those from 10 to 19 inclusive constitute the second decade All of the twenties make up the third decade, all of the thirties the fourth, etc. If one subtracts, for example, 8 from 13 and obtains the answer 5, one passes from the second decade into the first. If one subtracts 8 from 53, one passes from the sixth decade into the first. Likewise in addition, if one adds 7 to 16 and gets 23, he passes from the second decade into the third. All such cases involve what is called "bridging." A great deal of our adding and subtraction is done, of course, within the same decade Thus, 6 and 13, 18 minus 15, 35 and 4, 48 minus 45, and the like do not involve bridging

It is worth while, therefore, to know which of the 1237 combinations involve bridging The following is a summary of them

Among the 100 simple addition combinations, the 45 whose sums are between 10 and 18 inclusive involve bridging Among the 225 higher-decade additions, there are 90 combinations of this sort Of the 87 higher-decade additions which occur in multiplication and which do not occur in the 225 mentioned above, 26 involve bridging. Among the 100 subtraction combinations the 45 whose minuends are 10 or more are of this class Of the 175 additional subtraction combinations involved in

[1] The first reliable information on this point was furnished by Holloway, *op. cit*, pp 40, 41, 61, 73 Holloway's results were verified and extended by J H Smith ("Arithmetic Combinations," *Elementary School Journal*, xxi, pp 762–70) See also Counts, G S, *Arithmetic Tests and Studies in the Psychology of Arithmetic* Supplementary Educational Monograph, no. 4, University of Chicago Press.

short division, there are 60 of this sort Hence of the important and useful combinations in addition and subtraction the following only involve bridging:

In simple addition	45
In higher-decade addition	116
In simple subtraction	45
In the additional subtraction involved in short division	60
Total	266

Multiplication and division. Two further facts concerning relative difficulties are quite well established. Of the 100 simple multiplication combinations, those whose products are more than 25 are more difficult, generally speaking, than those whose products are 25 or less. In like manner, in short division those with dividends larger than 25 are more difficult than those whose dividends are 25 or less Of the former there are 32 which are more difficult Of the latter 112 are more difficult [1] This total is 144 These, combined with the 266 in addition and subtraction, yield 410 combinations which are very useful and rather difficult Constant drill upon these combinations will be needed even in the eighth grade To neglect this means certain failure to attain the desired degree of accuracy.

The Combinations in Various Settings

The foregoing represents a minimum requirement for the number of facts to be taught in the fundamentals of arithmetic To teach these and stop, however, would not be sufficient School children will also need the help of the teacher in the application and use of these facts. Some of these applications are so useful to people in general as to merit special mention here The teacher will

[1] See Appendix for complete lists

need to pay particular attention to carrying in addition, borrowing in subtraction, compound multiplication, and long division All of these are so well known that explanation of their nature is not needed It may be worth while, however, to say that "compound multiplication" is a term used to include all cases in multiplication where there is more than one figure in the multiplier.

The minimum amount which should be taught concerning these applications has already been described in part It is summarized here in a more complete form and in the order of importance It has been suggested that column addition should be limited very largely to columns whose sum is 39 or less This means that the child must learn accurately and carefully how to carry one, two, and three — and this is also the order of importance in carrying. It is much more important for a child to be able to add and carry one correctly than for him to be able to carry two, three, or more with accuracy This is true because most of the column addition which people use involves the carrying of one rather than that of more than one. Teach the children first the mastery of exercises where one is carried. When these are mastered, teach exercises where two and then three are carried. Then review all three frequently

In subtraction we never borrow more than one and this borrowing is done only in connection with the 45 subtraction combinations mentioned on page 23 Teach each of these 45 particular cases very carefully

In compound multiplication it is recommended that children be taught the mastery first of such exercises as have two figures in the multiplier and two in the multiplicand When this type of exercise is mastered, take up exercises that have two figures in the multiplier and three, four, five, etc., in the multiplicand Then pro-

ceed to exercises in which there are more than two figures in both multiplier and multiplicand

In long division a similar order is advisable. Teach first a mastery of the exercises in which there are two figures in the divisor and one in the quotient. Include answers with remainders also. Then proceed to exercises with two figures in the quotient, then to those with three in the quotient. After the pupil becomes proficient in all these exercises and not before, start him upon examples which have three figures in the divisor and one in the quotient, then upon those with two, three, etc. in the quotient keeping a three-figure divisor. This process may be continued as long as time permits, but it is not advisable to use examples which involve products or dividends which are extremely large. The question of how to deal with the applications of the fundamental number facts will be considered further in Chapter VII.

In concluding this chapter we may summarize by saying that we should teach first and most carefully that which is most useful and that which is most difficult. We should stick to each of the essentials until the child masters it. This is the best way to secure economy of time and effort in the long run.

CHAPTER IV

THE NEEDS OF THE PUPILS

In the preceding chapter much was said concerning what the child needs to learn in the fundamentals of arithmetic Two principles were enunciated which serve as guides to the needs of all children First, teach that which is most used or should be most used in life, and, second, suit the amount of drill to the difficulty of the task In this chapter I propose to consider the needs of pupils in a more individual and detailed manner Such specific information has been entirely lacking until very recent years. The sum total of human knowledge in print concerning detailed needs in the fundamentals of arithmetic is indicated by the references in the footnotes of this chapter. The meagerness of our information has been due in the main to two causes It has been felt that the difficulties of small groups of pupils are not worth considering and that such difficulties are distributed entirely by chance

Minutiæ in Teaching

It is quite certain that these two assumptions are responsible for the hopeless "doldrums" into which we have fallen in recent years in our efforts to improve results in arithmetic Big and striking things naturally attract attention. Furthermore, it is necessary to deal first with general principles and with material which is capable of wide application It is unwise to devote much attention to the five or ten per cent of pupils who have special needs when the ninety or ninety-five per cent are not properly cared for. We now have a voluminous literature dealing

with the method of teaching the fundamentals of arithmetic to the average child, and perhaps even to ninety per cent of the children The large elements have received attention in the past. But the hope of improvement now lies in the consideration of smaller elements. A slight increase in the efficiency of a machine may change failure into success. The removal of a very small amount of friction may increase efficiency enormously. It has been a mistake to overlook the minutiæ of learning and teaching, just as it would be a mistake to overlook the minutiæ which contribute to the success or failure of an automobile There is a very great difference between *almost* getting through the mudhole and actually getting through when we are driving our cars; yet the actual difference in the motive power may be very slight, indeed In teaching, the matter of minutiæ is even more important

Suppose a teacher has a class of thirty pupils, and suppose that all of the pupils are doing well save one At first sight it may seem that the failure of the one is a matter of small consequence The truth, however, is often quite the contrary Our best teachers are usually conscientious, and many of them often devote more effort and nerve energy to the one slow pupil than they do to all of the remaining twenty-nine put together. Meanwhile the twenty-nine are often compelled to mark time until the one slow pupil catches up, if he ever does They learn laziness and idleness, not arithmetic. Thus the efficiency which at first sight seems extremely high may degenerate into something little better than mockery.

A Study of Errors

Evidence showing the futility of the assumption that individual needs are distributed according to chance will

be presented later on in this chapter The method of approach used in collecting this evidence was the study of the individual errors of 3600 children This also proved an effective means of dealing with the small group of retarded pupils whose presence often plays such havoc with the actual results obtained by the teacher. In basing suggestions for the improvement of instruction upon error studies, it is assumed that success frequently results from the study of failures Industrial concerns find it profitable to study the situations in which their output falls short of the expected amount. Few salesmen talk about the weak points of what they sell, but it is safe to assume that all good salesmen have a pretty definite knowledge of what those weak points are Teachers, however, are almost always destitute of such information. There is an abundant literature and a host of correlation coefficients based upon the *correct* responses of the child, but no one knows the correlations which exist among the annoyances, distractions, and disturbances to which both children and teachers are subject.

Previous Error Studies

Wherein do children err and why? Our knowledge is far from complete, but enough is known to merit a presentation here. Most studies of errors have been based upon some standard test. Theisen in *The Janesville Survey*[1] describes an error study based on the Woody Tests in Arithmetic. C. J. Anderson had previously made a study of errors in division based on the same test[2] Uhl has also done similar work in column ad-

[1] Theisen, W W, and others *An Educational Survey of Janesville, Wisconsin* Madison, Wisconsin State Department of Public Instruction, 1918. 329 pp.

[2] Anderson, C J. "The Use of the Woody Scale for Diagnostic Purposes," *Elementary School Journal*, 18. 770–81. (June, 1918.)

dition based on the Courtis Practice Tests.[1] Counts made a much more complete study in 1917 based upon the Cleveland Survey Tests[2] These investigators proved one thing of great significance They showed that many errors are not of the chance type and made it appear doubtful if any of them are.

The most complete study of errors which has yet been made was carried on in Wisconsin in 1920–21 under the direction of the author This study would have been utterly impossible had it not been for the hearty cooperation of Wisconsin school superintendents, supervisors, and teachers These busy people spent hours of their time in the tedious and laborious process of listing the wrong answers obtained by their pupils on the Woody-Theisen Parallel Tests in Arithmetic[3] Copies of this survey were sent out in mimeograph form in December, 1921. The following is a summary of the findings:

Including duplications, 21,548 wrong answers were reported, of which 6107 were in addition, 6459 in subtraction, 4551 in multiplication, and 4431 in division The number of children involved ranged from 3044 in division to 3943 in addition. The schools represented were in various parts of the State and in varied environments Much of the data obtained relates to material beyond the scope of this book Only that portion of the report which relates to the fundamental processes is given in detail here. The remainder is briefly summarized

[1] Uhl, W L "The Use of Standardized Materials in Arithmetic for Diagnosing Pupils' Methods of Work," *Elementary School Journal*, 18 215–18 (November, 1917)

[2] Counts, G S. *Arithmetic Tests and Studies in the Psychology of Arithmetic* Chicago The University of Chicago Press, 1917 127 pp (Supplementary Educational Monographs, vol 1, no 4)

[3] Theisen, W W *Theisen-Woody Parallel Arithmetic Tests Addition, Subtraction, Multiplication, and Division, Grades 5–8* Madison, Wisconsin The Parker Company The parallel tests were made to follow the Woody Tests, Series A, as a second test.

All errors are included even when made by pupils who were unquestionably beyond their depth This was done on the theory that what a child does when he meets a new situation is the same as what he does when he meets one that he has totally forgotten.

Errors Found in the Course of the Wisconsin Study

Addition The following are some of the more important details in addition. Sixteen children in grade three and 9 in grade four got an answer of 90 to the problem $\begin{array}{r}70\\ \underline{29}\end{array}$ because they did not know the combination 9 and zero Twenty-four children in grade three, 27 in grade four, 23 in grade five, and 14 in grade six got an answer of 4 to the problem $4 + 1 =$ because they did not know the meaning of the plus sign Sixteen children in grade three got an answer of 15 to the problem $47 + 22 =$ (They said 4 and 7 and 2 and 2 equal 15) On five problems in column addition the children in grade three gave 100 wrong answers; those in grade four gave 82; and those in grade five gave 12. The total number of children tested was 597 in grade three, 674 in grade four, 703 in grade five, and 624 in grade six.

Subtraction In subtraction, 19 pupils in grade three and 10 in grade four got an answer of zero to $\begin{array}{r}5\\ \underline{0}\end{array}$; 10 in grade three got 1 for $\begin{array}{r}1\\ \underline{1}\end{array}$; 19 in grade three got 17 for $\begin{array}{r}11\\ \underline{8}\end{array}$, 19 in grade three got 14 for $\begin{array}{r}15\\ \underline{9}\end{array}$; 19 in grade three and 19 in grade four got 35 for $\begin{array}{r}39\\ \underline{5}\end{array}$; 57 in grade

three got either 10 or 11 for $\begin{array}{r}17\\8\\\hline\end{array}$, 125 in grade three, 104 in grade four, 38 in grade five, and 10 in grade six got either 30 or 35 for $\begin{array}{r}60\\35\\\hline\end{array}$, 47 in grade three got either 120 or 180 for $\begin{array}{r}470\\390\\\hline\end{array}$, 28 in grade three, 19 in grade four, and 13 in grade five got either 243 or 137 for $\begin{array}{r}675\\438\\\hline\end{array}$; 38 in grade three, 75 in grade four, 64 in grade five, and 10 in grade six gave typical wrong answers to $\begin{array}{r}1000\\729\\\hline\end{array}$. The total number of children tested was 549 in grade three; 623 in grade four, 650 in grade five, and 622 in grade six An inspection of these wrong answers shows three causes: (1) ignorance of zero combinations, (2) subtracting a small digit in the minuend from a larger one in the subtrahend, and (3) failure to borrow correctly

Multiplication In multiplication, 25 pupils in grade three and 12 in grade four gave either 314 or 1244 for $\begin{array}{r}310\\4\\\hline\end{array}$, 25 in grade three, 12 in grade four, and 11 in grade five gave 205 for $\begin{array}{r}40\\5\\\hline\end{array}$; 12 pupils in grade four and 11 in grade five got 9408 for $\begin{array}{r}1076\\8\\\hline\end{array}$; 12 in grade four got 37,168 for $\begin{array}{r}6078\\6\\\hline\end{array}$, 19 in grade four and 11 in grade five got 9455 for $\begin{array}{r}155\\60\\\hline\end{array}$, 12 in grade four got 3780 for $\begin{array}{r}135\\208\\\hline\end{array}$; 10 in grade three got 292 for $\begin{array}{r}26\\242\\\hline\end{array}$. The total

number of children tested was 467 in grade three; 642 in grade four; and 637 in grade five The source of trouble in nearly every case was reacting to zero in the multiplier and multiplicand as though it were 1. In the last exercise the pupils multiplied by the first figure of the multiplier and added the other two.

Division In division, 13 pupils in grade four got 8 for $4\overline{)36}$ The same number in the same grade got 1 for $6\overline{)1}$. Forty-four pupils in grade four and 42 in grade five got either 3 or 18 for $6 \div 3 =$; 19 pupils in grade four and 8 in grade five got 5 for $5\overline{)0}$, 19 in grade four and 16 in grade five got zero for $2\overline{)2}$, 37 in grade four, 25 in grade five, 9 in grade six, and 24 in grade seven got either zero or 16 for $4 \div 4 =$, 33 in grade five and 24 in grade six got either 403 or 43 for $14\overline{)56042}$ The total number of children tested was 597 in grade four, 641 in grade five, 557 in grade six, and 630 in grade seven The causes of the trouble were ignorance of division combinations, lack of acquaintance with the division sign, and trouble with zeros in the quotient.

Type Errors

The foregoing data have been presented in detail because they prove conclusively that there are very definite and very persistent type errors When children in school systems scattered over a whole State get the same wrong answer to the same problems, it is idle to speak of these errors as chance errors. The per cents of type errors in the addition of integers in grades three, four, and five was 17, 15, and 4 respectively In subtraction the corresponding per cents were 45, 25, and 9, and in multiplication they were 24, 13, and 3. In division the per cents of type errors in grades four, five, and six were 23, 14, and 4 respectively.

The remainder of the errors were of such a nature that their cause could not be ascertained from the test papers alone Further investigation since the time of this survey indicates that they are due in large measure to a varied combination of the same causes which produced the errors described in detail above. Even if such were not the case, the fact remains that by removing four or five causes, from 3 to 45 per cent of all the errors would be removed. For children who make these type errors we can guarantee the amount of improvement which will result when the causes of the errors are removed The question of how to remove the causes will be considered in Chapter VI

The per cent of type errors would have been much larger if another sort of test had been used in the survey The Woody tests aimed to present only one or two exercises representing each type of difficulty Some of these were presented in such a form that it was impossible to ascertain the cause of the error. This was not so much the case in subtraction, and as a result the total per cent in that operation was much higher. Evidently, before the final word is spoken concerning the nature and frequency of type errors, a great deal more work will have to be done.

An Error Study in Long Division

One type of procedure designed to give a more accurate notion of the individual and specific needs of pupils is given below. The data were collected in an effort to ascertain the nature and distribution of the errors which children make in long division. The following test was used.

To the pupil · This ladder has easy problems at the bottom and more difficult ones at the top How far can you climb on it the *first time* you try? Work each problem through *once.*

Begin at the bottom and work upward *Do not check your work.*

1 5097 ÷ 64
2 5716 ÷ 68
3 2809 ÷ 72
4 4019 ÷ 41
5 1519 ÷ 31
6 4209 ÷ 61
7 6914 ÷ 72
8. 4539 ÷ 72
9 1322 ÷ 32
10 5549 ÷ 72
11 4035 ÷ 72
12 5569 ÷ 64
13. 2549 ÷ 48
14 3696 ÷ 71
15 4388 ÷ 51
16 1808 ÷ 42
17 5600 ÷ 70

These exercises are constructed in such a manner that it is fairly easy to tell what caused the wrong answers. The test was given to 500 children in rural schools and in two cities. The total number of errors was 716 The errors were found to be grouped as follows

	Per Cent
Errors in estimating the quotient	39
Errors in multiplication combinations . .	21
Errors in subtraction combinations . .	21
Errors in carrying, borrowing, copying, bringing down, and failure to complete the exercise.	19

An inspection of these per cents leaves little doubt as to the character of the needs of these particular children. Nearly half of their trouble lies in multiplication and subtraction rather than in division — indeed, more than half of it if carrying and borrowing are included Now, multiplication and subtraction are presumed to be prerequisite to long division, yet here we find teachers try-

ing to teach children to divide when at least half of the difficulty lies in what precedes division.

Type Errors are Widespread and Constant

Having reached this point in our study of errors, it seemed advisable to stop and devote our attention to a means of removing the errors whose nature and extent were already known rather than to spend more time in the search for further types. When our remedial program has been completed up to this point, we hope to be able to devote further time to a study of additional causes. One thing is encouraging A comparison of error studies shows that type errors are remarkably widespread and constant Theisen, in his study of the errors made by fifty children at Janesville, found nearly all of the type errors which were found later in the responses of more than three thousand children scattered over the entire State of Wisconsin. Similar results have been obtained in several studies which have since been carried on in local areas Now, a study of the errors made by a thousand or more children can be carried on only with the cooperation of many people On the other hand, a study of the errors of fifty or a hundred children can easily be made by one person The fact that such a study reveals typical errors gives much significance to the work of individual investigators who are dealing with small groups. Progress, therefore, should be easy in the future.

Counting on the Fingers

Two more facts remain to be mentioned concerning the needs of children in the fundamentals of arithmetic, as revealed by the foregoing studies The first relates to a peculiarity of the errors of children in simple addition. Lists were made out of the wrong responses

which children gave to such combinations as 9 and 6 From this it appears that most children make the correct response Of those who err, a large number will respond 14 and about an equally large number will say 16. Two other equal but smaller groups will respond either 13 or 17. This is undoubtedly very largely due to counting with the fingers, tongue, or some other portion of the body. It is one of the chief arguments against permitting children to indulge in such a practice. To some extent the same thing is discernable in the other processes.

Interference in Learning

The second fact is that of interference. Large numbers of children substitute something which they have previously learned for the thing which is required of them. Those who are asked to add will multiply; those who are asked to divide will subtract This is one of the worst of all errors, and it seems to grow worse as the child progresses from grade to grade. It would not seem to be difficult for any one to learn that 2 times 3 is 6, yet children continually give 5 for an answer, because they have learned 2 plus 3 so well. In like manner there is a very marked tendency to respond "zero" to any number divided by itself. Another common type of interference is the confusion of one multiplication combination with another. Thus we find $3 \times 2 = 3$, $4 \times 4 = 27$, $7 \times 2 = 16$, $6 \times 7 = 35$. In fact, the wrong response is *nearly always* some other product of the multiplication table

Informal Reports on Errors

Just as the constant and general nature of errors tends to lend weight to the results of individual investigators, so the reports of teachers are worthy of considerable

attention even when they have not made a formal survey The following errors are reported by teachers often enough to insure for them a place in this chapter:

(1) Failure to put zeros in the quotient when necessary, as in $7\overline{)2863}$ The wrong answer is 49

(2) Errors arising when children are required to add a number between 5 and 10 to one less than 5 For example, $2 + 8$ is more difficult than $8 + 2$ This seems partly due to the habit of counting on the fingers or tapping In $8 + 2$ the child needs to tap only twice, which is easy; but in the case of $2 + 8$ the child has to tap eight times and the chances for error are much increased.

(3) In higher-decade addition, when the next number in the column is the same as the digit in units place in the preceding sum For example, in the exercise

$$\begin{array}{r} 27 \\ 45 \\ 78 \\ \underline{34} \end{array}$$

suppose the child begins adding at the bottom of the units column When he gets up to the number 7, he should have 17 in mind The addition of such numbers as 17 and 7, 26 and 6, 38 and 8, etc , seems rather difficult This is probably due to too much expectation of transfer. The children have been taught 7 and 7, 6 and 6, and 8 and 8, and have found these combinations rather easy because they are reenforced by some things learned in multiplication Such is not the case, however, in the corresponding higher-decade additions

Errors in Arithmetic Reasoning

The preceding pages of this chapter have been devoted to the needs of children in the fundamentals of arithme-

tic as disclosed by a study of errors During 1922 the same method was used in order to discover the weak spots in arithmetical reasoning. The results are still incomplete, but the following classification of errors is available based upon 30,000 errors made in the Buckingham Problem Test[1] by 6000 children in eighteen counties and one large city:

	Per Cent
1. Total failure to comprehend the problem	30
2. Procedure partly correct, but with the omission of one or two essential elements	20
3. Failure to respond to fundamental quantitative relations	10
Total	60
4. Errors in fundamental processes	20
5 Miscellaneous errors	2
6. Errors whose causes could not be discovered	18
Grand Total	100

Summary

With reference to the needs of pupils in the fundamentals, the following is a summary of the typical sources of error set forth in the foregoing chapter:

1 Trouble with zero combinations in each of the four processes.
2 Failure to deal with number facts when presented in the equation form
3. Difficulties in column (higher-decade) addition.
4. Trouble in subtraction when a digit in the subtrahend is greater than the digit just above it in the minuend.

[1] Buckingham, B R *Scale for Problems in Arithmetic* Bloomington, Illinois: Public School Publishing Company

5 Interférence between what is required and what is already known (harmful transfer).
6 Ignorance of the combinations in all of the processes.
7 Estimating the quotient in long division.
8 Carrying in addition
9. Carrying in multiplication.
10 Borrowing
11 Copying.
12. Bringing down in long division.
13 Failure to complete the exercise even when the time is sufficient.

In connection with the typical responses of pupils it has also been shown in this chapter that small errors and errors made by only a few pupils have a cumulative effect which is very pronounced. The existence of chance errors is doubtful.

CHAPTER V

HOW THE TEACHER CAN DISCOVER THE NEEDS OF PUPILS

THE title of this chapter assumes that teachers both *can* and *should* discover the causes which are operating to retard the progress of their pupils. The truth of this assumption is usually acknowledged in theory, but it very frequently fails to carry over into actual practice. Because of this, the teacher is seldom considered to be an expert by the general public We often hear of expert mechanics, expert physicians, and expert lawyers, but little is said outside of pedagogical books and articles concerning expert teachers An essential characteristic of an expert is his ability to diagnose difficulties better than ordinary people can. Thus the expert mechanic must know how to tell what is wrong with a machine that will not work. An expert physician must be able to tell what is the matter with people who are ill And an expert lawyer must be able to point out the weak points in his opponent's argument In like manner, expert teachers must be capable of making an educational diagnosis of pupils who are not succeeding in their school work as well as they should The teacher must know better than any one else how to discover the nature and causes of the disabilities under which pupils labor

The past century has seen a development and application of the principles of scientific method to an extent unheard of before Advancement in scientific research has made possible the great inventions which have so recently transformed our mode of life. A similar advance

has also been made along professional lines. Together with the growing multiplicity and complexity of our ways of doing things, has come of necessity a multiplicity and complexity of technical knowledge and skill. Much of this knowledge is available in books, but the place to learn how to apply it effectively is in the presence of actual situations. The machinist has become an expert by an extended experience in making and repairing complex machines, the surgeon by the actual performance of intricate surgical operations; and the physician by continual practice with people who are ill. Thus the growth of modern thought has made necessary more and more specialized forms of expert science.

The Need for Expert Educational Diagnosis

The teaching profession has also felt the impulse of this movement The growth of technical information relating to methods of teaching had developed to such a point seventy-five years ago that it was found necessary to provide normal schools for the professional training of teachers. These institutions have done a valuable and extensive work, but their influence is not yet what it should be In many respects they have lagged behind other institutions which give professional training This is particularly true in their failure to equip their students with a mastery of scientific methods or even with a scientific attitude of mind. If an automobile fails to work, one can find a man in any country garage who can usually make a satisfactory scientific diagnosis of the difficulty and a reasonably good prescription of the treatment necessary In like manner, if a person becomes ill, one can usually find a physician who can make a diagnosis and prescribe a remedy in accord with scientific procedure. But how is it when Mary and Johnnie fail to make their

grade in school? Most teachers, indeed, will offer a diagnosis They tell us that Mary is restless and giddy and refuses to think about what she is doing, that Johnnie is indifferent and bad and so disgustingly careless and slovenly in his work. This is diagnosis of a sort, but it certainly is not scientific and it is quite likely to be faulty. Any school patron could do as well Hence we as teachers have not yet been able to convince the public that we are experts in our business.

The main cause of the trouble lies in our ignorance of the disabilities and weaknesses which beset our pupils. We have not gone about our work in a scientific manner. The machinist can tell the general disabilities to which machines are subject and he judges from the behavior of a machine which particular disability is present The doctor, in like manner, looks us over, observes our behavior, and then from his general knowledge of diseases identifies the thing which is afflicting us There was a time when doctors did not have sufficient knowledge of diseases to identify any particular one. Then they met the situation by saying that some evil spirit had gotten into the patient, just as teachers often say that Mary and Johnnie are possessed by the spirit of laziness, stubbornness, or plain meanness. Doctors, however, have learned that diseases are caused, not by spirits, but by certain definite and specific conditions. They have learned that there are many diseases They have identified these diseases and have given them names They have learned that the behavior of the patient is the key to his disease They know that specific diseases require specific treatment, and no longer try to cure typhoid fever and rheumatism by hocus-pocus or incantation.

Teachers, too, must learn how to identify the educational diseases which afflict pupils. Many of these dis-

abilities are so vague as yet that we have no names for them. We do know, however, that they are specific. Mere efforts to make the work interesting or to compel the child to study will not remove them any more than charms and amulets will cure scarlet fever We are beginning to learn, too, that the behavior of the pupil affords the symptom of his educational disability just as fever, cold feet, and a peculiar-looking throat are symptoms of influenza

The purpose of this chapter is to set forth certain types of pupil behavior which are symptomatic of some of the most widespread educational disabilities, with the hope that the teacher on the job may be able to use them profitably in the educational diagnosis of her pupils.

Two Types of Educational Disabilities

There are two general types of disabilities, one that must be outgrown if it is ever removed, and the other that may be removed with comparative ease when its nature and causes are known The former is the result of a lack of mental ability on the part of the child, while the latter is due to poor teaching, poor health, and the like It is evidently of great importance for the teacher to have some means of discovering which type is responsible for the child's shortcomings If the former type of disability is present, the teacher is not to blame, if the latter, she and her predecessors are at least partially at fault

How can the teacher tell whether or not a lack of mental ability is the cause of the pupil's failure?

The use of an intelligence test is the best help. Consider the grade which the child is in and the average age of the pupils who are generally found in that grade. Suppose you are dealing with a child in the fifth grade,

for example Fifth-grade children should be from ten to eleven years of age. But suppose this pupil responds to an intelligence test just about like a seven-year-old child should respond This means that your fifth-grade child has the *mental age* of a seven-year-old child regardless of how old he may be. Such a pupil would have only second-grade ability, and it would be a mistake to try to teach fifth-grade subject-matter to him Efforts to do so, if vigorously carried out, would tend to subject the child to overstrain The teacher is not to blame under such circumstances when the child fails On the other hand, she is to blame if she robs the other pupils of attention and tries to get one or two children through when they haven't the ability to do the work even under the best teacher that ever lived.

It is a mistake, however, to rely upon one intelligence test alone. Some children are so timid and nervous that they cannot do themselves justice on one such test. There are a few children, too, who may receive low scores on an intelligence test when they are in reality good in arithmetic. Thus a pupil may have more than fifth-grade ability in arithmetic and still get a low score on an intelligence test because he has almost no ability in anything but arithmetic In other words, the exercise of common sense and the use of other data than those derived from intelligence-testing are essential

While the use of an intelligence test is advisable, there are many cases when it will be impossible because of the expense involved or because the teacher does not know how to give such tests. Under these circumstances, the following "symptom" is worth remembering. If you find a child in grade three or above who simply cannot remember the arithmetic combinations — if he *always* forgets by to-morrow what you teach him to-day — the

chances are that you are dealing with a pupil who is lacking in mental ability. In that event, give the pupil *his share only of your time, but do not neglect the other pupils on his account*

Two things further should be said in this connection First, if a child is mentally incapable of doing what is expected of him *now*, do not assume that he never will be able to meet the requirements Some babies say *da-da* before others do, but none of them does so until he is fully ready. In like manner, the children who cannot possibly do fifth-grade work this year may do it well next year Mental growth is rapid in some and slow in others With a few individuals it may be slow for a long period and then it may take place rapidly When the baby refuses to say *da-da*, sensible parents merely wait until he is ready. Teachers and parents must often do likewise with the slow child who cannot learn long division or fractions.

Second, some children are capable of doing the work in the grade in which they are placed, but they learn so slowly that the teacher, in justice to the other pupils, cannot afford to take the time required to teach them thoroughly The only thing for the teacher to do in such a case is to tell the child's parents very clearly how matters stand and recommend that they employ a tutor for him. If they fail to do so, do the best you can for the child, but do not rob the other children of their share of your attention Do not try to teach that which the children cannot learn even with the best of teachers, and do not teach slow pupils at the expense of other pupils who can profit more from your attention.

Symptoms of Disabilities in Arithmetic

The second type of disabilities is due in large measure to the environment and experience or lack of experiences

which the child has had The best approach to a diagnosis of these abilities is through the observation of types of behavior which are symptomatic In the case of arithmetic, the teacher will need to know what the child *does* and *how he thinks* when confronted by a problem. It is often possible to discover disabilities by merely examining the wrong answer which the child gets to his problem Some of these significant errors have already been noted in Chapter III, together with a list of the most common disabilities. Additional samples are given here as a further aid to a teacher who is a beginner in educational diagnosis.

How to Diagnose Errors in Addition

Simple addition.

(1)	(2)	(3)
4	5 + 1 = 5	70
2		29
8		79

The trouble occurs in connection with 2 and 4, 5 and 1, and 2 and 7 respectively It is possible that the child does not know these combinations. Other possibilities are that he multiplied instead of adding in the first case, and that he confused the plus and multiplication signs in the second. In the third he may have attempted to add by tapping. After making the seven taps, he may have written "7" instead of "9" The second group of possibilities are the more probable.

Column addition.

(1)	(2)	(3)	(4)	(5)	(6)
386	32	13	32	32	13
197	36	34	36	36	34
174	15	22	15	15	22
389	81	70	73	113	79
1136					

In the first, the error is in the tens column, and there are three possibilities, mistakes in combinations, having carried 1 instead of 2, having written 3 instead of 4 after obtaining 34 as the sum of the tens column. The first and third of these are the more probable. In the second example, the error lies either in the combinations or in a failure to see the number at the top of the units column. The odds are about even between the two In the third example, the trouble undoubtedly lies with the combinations. In number four, the error may be due to mistakes in combinations or in failure to carry The latter is the more probable In example five, the trouble is due either to combinations or to having carried 4 instead of 1. The former is the more probable

In number six, the child either made a mistake in the combinations or carried 1 when there was nothing to carry The latter is the more probable.

Addition of common fractions.

(1)	(2)	(3)
$\frac{1}{3}+\frac{1}{2}=5$	$\frac{1}{8}+\frac{1}{2}+\frac{3}{4}=1\frac{1}{4}$	$\frac{1}{3}+\frac{1}{2}=\frac{1}{6}$
(4)	(5)	(6)
$\frac{1}{5}+\frac{1}{5}=\frac{2}{10}$ or $\frac{1}{5}$	$\frac{1}{3}+\frac{1}{2}=\frac{2}{6}=\frac{1}{3}$	$\frac{1}{5}+\frac{1}{5}=\frac{1}{10}$

In the first example, the child neglected the numerators, added the denominators, and wrote the result as an integer In the second, an error was made in adding the numerators after each fraction had been reduced to a common denominator In the third, the child multiplied. In the fourth, he added the numerators for a new numerator and the denominators for a new denominator. In the fifth, he added the numerators for a new numerator and multiplied the denominators for a new denominator. In the sixth, the last process was reversed.

Addition of decimals.

73.0205 + 101 3. + 35 + 87 26 + 49 006 = 788985

The child copied these numbers in a column so that the ends were even *on the right,* added correctly, but forgot all about the decimal point. This is a very frequent error

Addition of denominate numbers.

(1)	(2)
6 ft. 3 in	6 ft. 3 in
1 ft 7 in.	1 ft. 7 in.
3 ft 6 in.	3 ft 6 in.
11 ft. 6 in.	10 ft. 16 in.

In the first case, the child added as though he were working with abstract units. Having obtained 16, he wrote down the 6 and carried the 1 In the second case, no effort was made to reduce the answer to proper terms These two types cover nearly all of the errors found in the addition of denominate numbers Sometimes they are mixed with errors due to a lack of knowledge of the addition combinations, and in rare cases we find that trouble with the addition combinations is the sole cause of the error

How to Diagnose Errors in Subtraction

Simple subtraction

(1)	(2)	(3)	(4)	(5)
8	16	13	11	16
5	9	8	7	9
7	14	4	5	25

(6)	(7)	(8)	(9)	(10)
6	6	4	9	7 − 4 = 7
0	0	4	3	
0	5	1	3	

Each of these shows an ignorance of subtraction combinations, though number ten may be due to a lack of acquaintance with the minus sign In both addition and subtraction, when children miss a combination they miss the answer by just one unit more often than they make any other error This is shown in numbers three and four. It seems to be due to counting on the fingers.

Numbers six and seven are samples of the very common trouble which children have with zero combinations. In number seven the child does not feel that he has subtracted unless his answer is less than what he started with. Number eight shows a rather frequent trouble that children have in subtracting a number from itself. In numbers six and nine the error may be due to the substitution of another process for subtraction.

Borrowing.

(1)	(2)	(3)	(4)	(5)
50	50	21	279	393
25	25	9	190	178
35	30	29	129	225

(6)	(7)	(8)	(9)	(10)
1000	1000	1000	21	393
537	537	537	9	178
1537	563	1000	2	115

Numbers one and two are very common in the lower grades and are found in all grades The error in number one may be due to a failure to "pay back" what is borrowed, or it may result from something like the following line of thought "She wants me to take 5 from zero. I don't know how to do it, but we learned not long ago that zero from 5 leaves 5" This reversing of digits in the subtrahend and minuend is very common In number two the child probably thinks: "If you start with zero

and take 5 from it, you certainly would still have zero You can't get anything by taking something from nothing "

In number three the child feels that he cannot take 9 from 1, so he uses 9 times 1 Number four shows a sure case of reversion Nine from 7 is responded to as 7 from 9 should be Number five is either a case of reversion or a failure to "pay back" what is borrowed The odds favor the former Number six might be addition, but much oftener it is reversion Number seven is a case of failure to "pay back" what was borrowed. Number eight is a case of "you can't take something from nothing and get anything" Number nine is a case of "paying back twice" what was borrowed Number ten is an example of where the child "pays back" when nothing has been borrowed.

Subtraction of common fractions.

(1)	(2)	(3)	(4)	(5)
$8\frac{7}{8}$	27	$8\frac{7}{8}$	$\frac{5}{12} - \frac{2}{10} = 7 - 8$	$75\frac{3}{4}$
$5\frac{3}{4}$	$12\frac{5}{8}$	$5\frac{3}{4}$		$52\frac{1}{4}$
$3\frac{4}{12}$	$15\frac{5}{8}$	$3\frac{4}{4}$		$23\frac{3}{4}$

A small amount of inspection shows what was done in each of the five cases above In number one, the denominators were added while the numerators were subtracted In number two, the fractional portion of the subtrahend was brought down Numerators and denominators were both subtracted in number three. Number four is an illustration of numerators subtracted from corresponding denominators. In number five the numerators were multiplied.

Subtraction of decimals

$$10 - 6.25 = 6.15$$

The wrong answer is due to an effort to put the number which is *apparently* smaller under the number that is

apparently larger so that the right-hand ends of the numbers will be even

Subtraction of denominate numbers

(1)	(2)
5 yd 1 ft 4 in	5 yd 1 ft 4 in.
2 yd 2 ft 8 in	2 yd 2 ft 6 in
3 yd 1 ft 4 in.	2 yd 8 ft 8 in

In the first case, the subtraction is reversed, in the second, the subtraction is made as if the problem required the subtraction of 226 from 514.

How to Diagnose Errors in Multiplication

Simple multiplication

(1)	(2)	(3)	(4)	(5)
8766	6078	379	8 7	8 7
8	6	05	4	4
69128	36608	15 95	36 2	40.8

(6)	(7)	(8)	(9)	(10)
310	8766	7889	529	135
4	8	9	7	208
314	6928	70001	3712	1080
				135
				270
				29430

The wrong answers shown in numbers one, two, and nine are most likely due to ignorance of combinations The pupils probably responded "8 times 8 are 63," "6 times 7 are 56," "7 times 9 are 72," respectively In number four, the possible errors are 4 times 7 are 48, or 4 times 8 are 34, or 32 and 2 are 36. One cannot tell with certainty from an inspection of the answer. In number five, the error is probably 4 times 8 are 40 with failure to carry In number six, the child said 4 times

o are 4, and brought down the remainder of the multiplicand. In number seven, the figure 7 is omitted from the multiplicand. In number eight, the child either said 9 times 7 are 62 or 63 and 8 are 70 The odds favor the latter. In number ten, the pupil used the zero of the multiplier as though it were a 1

Compound multiplication All the errors of simple multiplication are likely to occur also in compound multiplication. Thus, in

$$\begin{array}{r} 9833 \\ \underline{49} \\ 78497 \\ \underline{39332} \\ 472817 \end{array}$$

the child missed 9 times 9, and also failed to add correctly, having carried when carrying was not necessary.

Multiplication of common fractions.

(1)	(2)	(3)
$9 \times 5\frac{2}{3} = 15$	$3\frac{1}{2} \times 4\frac{1}{2} = \frac{63}{4} = 14\frac{3}{4}$	$\frac{1}{2} \times \frac{1}{2} = 1$

(4)	(5)	(6)
$\begin{array}{r} 798\frac{2}{3} \\ \underline{25} \\ 3990\frac{2}{3} \\ \underline{1596} \\ 19950\frac{2}{3} \end{array}$	$\begin{array}{r} 18 \\ \underline{2\frac{5}{9}} \\ 36\frac{5}{9} \end{array}$	$1\frac{1}{4} \times 8 = 12$

(7)	(8)	(9)
$\frac{1}{\cancel{6}_{2}} \times \cancel{3} = 2$	$\frac{1}{2} \times \frac{1}{2} = \frac{2}{4} = \frac{1}{2}$	$\frac{1}{6} \times 3 = \frac{1}{18}$

In number one, the answer is written backwards In number two, there is an error in division in reducing the answer In number three, the child confuses the multiplication sign with that of addition, while in number eight,

he adds the numerators and adds the denominators In numbers four and five, the child merely brings down the fractions. In number six, the pupils found $\frac{1}{4}$ of 8 to be 4 In number seven, the answer is inverted In number nine, the child divided.

Multiplication of decimals. All the errors of simple and compound multiplication play a part. The main additional difficulty lies in the failure to point off the product correctly.

Multiplication of denominate numbers.

(1)	(2)
$\begin{array}{r} 4 \text{ ft } 5 \text{ in} \\ 5 \\ \hline 20 \text{ ft } 25 \text{ in} \end{array}$	$\begin{array}{r} 4 \text{ ft } 5 \text{ in} \\ 5 \\ \hline 22 \text{ ft } 5 \text{ in.} \end{array}$

In the first case, the child failed to reduce the answer to proper terms. In the second, he acted as though he had been required to multiply 45 by 5

How to Diagnose Errors in Division

Division combinations

(1)	(2)	(3)	(4)
$\begin{array}{r} 5 \\ 1\overline{)6} \end{array}$	$\begin{array}{r} 12 \\ 4\overline{)36} \end{array}$	$\begin{array}{r} 16 \\ 4\overline{)4} \end{array}$	$\begin{array}{r} 1 \\ 1\overline{)6} \end{array}$

(5)	(6)	(7)
$\begin{array}{r} 0 \\ 1\overline{)6} \end{array}$	$6 \div 3 = 3$	$\begin{array}{r} 0 \\ 2\overline{)2} \end{array}$

Number six may also have been the result of a lack of acquaintance with the division sign.

Short division.

(1)	(2)	(3)	(4)
$\begin{array}{r} 682 \\ 7\overline{)4494} \end{array}$	$\begin{array}{r} 670\frac{4}{7} \\ 7\overline{)4494} \end{array}$	$\begin{array}{r} 661\frac{1}{2} \\ 8\overline{)5392} \end{array}$	$\begin{array}{r} 48 \\ 6\overline{)2448} \end{array}$

The errors in numbers one and two are due to an ignorance of division combinations. In number one, the child thought $29 \div 7 = 8$, in number two, $29 \div 7 = 7$ In number three, the error is 53 less $48 = 4$. In number four, the child failed to write zero in the quotient in response to " 6 into 4."

Long division The main new error is failure to estimate the quotient correctly This is not as frequent as errors in multiplication and subtraction when the latter are taken together.

```
   (1)            (2)            (3)
    358            39             392
25)9900        25)9900        25)9900
   75             75             75
   240            240            240
   125            185            225
   1120            65              50
    200

   (4)            (5)            (6)
     25             2 5/7           403
17)345         17)345         14)56042
   34             34
     5             05
```

Number one shows a failure to estimate the quotient correctly where the child thinks " 25 into 240 " Number two shows a failure to carry where the child multiplies 25 by 9 Number three shows an error in subtracting 225 from 240 Numbers four and five exemplify failure to deal with situations where the quotient ends with a zero followed by a fraction Number six shows another case of zero trouble

Division of common fractions.

(1) $\frac{2}{5} \div \frac{7}{2} = 1\frac{2}{5}$ (2) $\frac{2}{5} \div \frac{7}{2} = \frac{5}{7}$ (3) $\frac{2}{5} \div \frac{7}{2} = \frac{35}{4} = 8\frac{3}{4}$

Nearly all of the errors are connected with inversion In number one, the child failed to invert at all, in number two, he inverted both divisor and dividend, and in number three he inverted the dividend.

Division of decimals

(1)	(2)
$$\begin{array}{r} 1.2 \\ 2.2\overline{)26.4} \end{array}$$	$$\begin{array}{r} .12 \\ 2.2\overline{)26.4} \end{array}$$

Errors found in the division of whole numbers play a considerable part Almost all of the new errors are of the two kinds above shown In the first case, the child thinks he must put the point in the quotient *just above* where it is in the dividend. In the second case, he thinks he must point off as many places in the quotient as there are in the divisor and dividend taken together.

How to Diagnose Errors in Reasoning Problems

The most frequent trouble in this connection is the failure on the part of the child to know what process to use. He looks at his teachers and asks artlessly, "Do you subtract?" On examination or test day the teacher, of course, does not answer Under such circumstances the child acts like any normal but ignorant person would. He thinks "She wants me to do something. I don't know what it is, but I'll take a chance at it anyway" There are four fundamental processes, and the child has one chance in four in one-step problems of getting the correct answer by mere guess. He will guess wrong, however, about three times out of four Thus, in the problem, "We learn to spell two new words a day in our school How many new words do we learn in eight days?" We find by far the largest number of wrong answers to be 10, 6, and 4. These represent the wrong guesses. The

children add, subtract, or divide because they are *unlucky*. It follows, of course, that some of the children who get the right answer have guessed luckily. In problems of two or three steps the same thing occurs, but it is not so easily detected

The first thing to do in trying to account for wrong answers to reasoning problems is to find every possible way to "juggle" the figures. In more than five cases out of ten you will find in your list the wrong answer obtained by the child.

Another large group of errors arise from the fact that the child misreads or misunderstands some portion of the problem. Consider the following problems[1]:

(1) A boy had 210 marbles. He lost one third of them How many had he left? Pupil's answer, 70

(2) A coat cost ten times as much as a hat Both together cost \$66 00 How much did the coat cost? Pupil's answer, \$6 60

(3) Find the total cost of the following purchases: $8\frac{3}{4}$ yards of flannel at 96¢, $4\frac{1}{4}$ yards of braid at 16¢, 12 yards of embroidery at $22\frac{1}{2}$¢, 10 yards of lace at $27\frac{1}{2}$¢ Pupil's answer, \$1 62

(4) What is the product of 6 and 8? Pupil's answer, 14.

In the first problem, the pupil fails to *read what is required* He assumes that he is to find how many marbles were *lost* In the second problem, he leaves the cost of the hat entirely out of consideration In the third, he fails to understand that the prices quoted are prices *per yard* In the fourth, he does not know the meaning of the word *product*. He adds, just to be doing something.

A third type of error arises from an ignorance of fundamental relations, such as those of income and outgo,

[1] Problems (1) and (3) are taken from the *Buckingham Problem Test*.

profit and loss, velocity and time, and the like. For example:

(1) I wish to buy a house and lot I find a place that rents for \$720 00 a year The taxes and up-keep amount to \$240 00 per year How much can I afford to pay for the house and lot if I wish to clear 6 % per year on my investment?

Pupil's answer, \$16,000

(2) The distance from New York to Chicago is 980 miles A train starts from Chicago toward New York, running at the rate of 40 miles per hour. How long will it take the train to reach New York? Pupil's answer, 39,200 hours

In the first case, the pupil *adds* the amount spent on the property to the amount received from it. In the second, he *multiplies* the distance by the rate to find the time.

In addition to the errors just given and their symptoms, any teacher who investigates the matter will find errors in the children's work which are due to mistakes in the fundamental processes These should be diagnosed as suggested on pp 46 to 55

It will be noticed that in many cases the disability cannot be surely identified by the mere inspection of the children's work In some instances two or three possibilities have been given This means that upon a first inspection we may be unable to decide which of two or three things is the matter, just as the physician is sometimes uncertain as to whether we have the influenza or merely a bad cold. In both cases further diagnosis is necessary and possible. In all matters of doubt the teacher should request the pupil to do his adding, multiplying, and the like *aloud* [1]

This takes very little time and is very much worth while because it is almost sure to yield the true diagnosis. It

[1] See Uhl, *op cit*, p. 29

is recommended that the teacher attempt first to account for failure on the part of the child by the inspection of his wrong answers If doubt still remains, be sure to have him do the work aloud. Having thus gotten a correct notion of the true nature of the child's difficulty, it is usually easy to remove it.

Educational Inventories

The business man often makes an inventory in order that he may carry on his business more effectively In like manner the teacher will frequently need to make educational inventories. Such instruments are given in the Appendix On pages 155–78 are the things which every child needs to know. The teacher's task is to find out just how many of the required things each child knows and just what and how many he does not know. In obtaining this information the procedure is as follows, taking the 100 addition combinations on pages 155–56 as an example. Ask the pupils to write the numbers from 1 to 100 on their tablets, leaving a space at the right of each column. Read aloud the first combination in this manner: "How much is 2 and zero? Write what you think it is just after the number 1 on your tablet" See that all get it in the right space Then say: "I am going to read some more combinations to you. Please write the answers after 2, 3, 4, etc I am going to read them rapidly. You will have to hurry to keep up If you do not know the answer, skip the space and go to the next number." Read the combinations, then, "2 and 6," "3 and zero," and so on When a child does not know a combination well, or when he has to count on his fingers, he often spends too much time on it Often he finally gets it right, but in the mean time he has forgotten what the next combination is. In checking

papers it is well to remember, therefore, that a blank often represents trouble with the correct answer just preceding it

In order to help some children to get back on the track after they have been "lost," it is well to say, "Number ten (or whatever the number is) is 5 and zero." Then proceed as before Both teacher and pupil will need preliminary practice before the final try-out is made. The pupil will need practice to enable him to skip correctly when he does not know the combination. The teacher will need practice to enable her to read the combinations regularly and at the correct rate of speed. A reasonable time between the readings is.

Grade two	Six seconds
Grade three	Five seconds
Grade four	Three seconds
Grade five	Three seconds
Grade six	Two seconds
Grade seven	Two seconds
Grade eight	Two seconds

These limits should be gradually shortened if the test is used as a practice exercise The final limit is the speed with which the children can write the answers It goes without saying that the teacher should give no further help after the children understand what is to be done.

All of the combinations given on pages 155-72 can be tested out in this way. In this manner the teacher and pupil can learn just what each pupil already knows and just what remains to be learned *Be sure* that each child keeps a record of the combinations which he misses

A final word seems necessary to meet the objection which is sometimes raised against giving tests to children under time pressure It is said that children get nervous and excited and cannot do themselves justice. The an-

swer is, that we do not care if they do In actual life situations we do our adding, multiplying, and the like when we are thinking of something else We cannot afford to have to stop and think how much 9 and 6 are or how to spell our names. These things must be done *accurately* and almost *unconsciously*. Our task as teachers is to teach these combinations so well that the pupils will respond correctly *even when they are scared or excited.*

So far as the strain is concerned, the test does not last long enough really to injure even the most excitable pupil. One of the best things we can teach our pupils is the power to concentrate their attention, marshal their forces, and settle down to accurate, concentrated work for short periods Instead of harming pupils, it helps them and they enjoy it. Every practical teacher knows this.

Summary

Suggestions have been made as to locating the pupils who are so poorly equipped mentally that it isn't worth while to devote time or effort to teaching them in public schools Lists have also been given of the "symptoms" of special disabilities in addition, subtraction, multiplication, and division, and in reasoning problems. This part of the chapter is to be used mainly as reference in connection with individual case studies Finally, a method has been outlined for using as inventory instruments the lists given in the Appendix.

CHAPTER VI

NEEDFUL TYPES OF PRACTICE MATERIAL

In Chapter IV we learned what the needs of the pupils are in one very large area. Chapter V tells how a teacher may discover what her pupils need This chapter will concern itself with the type and amount of practice required to correct the weaknesses known to exist on a large scale.

Guiding Principles Relating to Practice

In seeking for needful types of practice material, there are four guiding principles. First, additional practice should be provided in that which is inherently more difficult for children to learn. The practice material should, therefore, be arranged in the proper order so that each element can be practiced when it is most needed. Second, additional practice should be provided for those elements in which the pupils are weak Third, practice material should be provided to take care of situations in which needful material does not now exist in printed form. We cannot afford to hold children responsible for something that they have never had a chance to learn. Fourth, that which is needed should be taught directly. To expect useful habits in arithmetic to be formed merely by transfer is not in accord with scientific principles and is sure to result in waste and disappointment.

Inherent difficulties These principles will be considered in the order indicated above. The question of inherent relative difficulty has already been discussed in Chapter III The facts brought out there are summarized here for convenience

In simple addition the combinations whose sum is more than 10 are relatively more difficult except those in which a number is added to itself, such as 9 and 9, 8 and 8, and the like. In the latter cases the difficulty varies almost exactly with the size of the numbers

Large numbers added to small numbers are usually more difficult than the reverse. Thus, 3 and 9 is more difficult than 9 and 3 In higher-decade addition, bridging generally involves additional difficulty. In subtraction, multiplication, and division much the same situation is found. The larger the numbers involved, the more difficult the combinations One important exception to this rule occurs in division where it is required to divide a number by itself For some children these combinations seem the most difficult of all At least they are responsible for most of the errors These facts are well established even when the pupils have had an equal amount of practice beforehand

Elements in which pupils are weak. In the study of errors described in Chapter IV, it was impossible to know how much previous practice the children had received. The results, however, were largely in accord with the facts given above. One outstanding condition, however, developed. The zero combinations proved very troublesome This was particularly true in subtraction, multiplication, and division.

Needful Practice Material Which Is not now Provided

The trouble with zeros raised the question as to whether or not sufficient practice material is provided for use in schools. In order to answer this question, the author checked through one set of the Courtis Practice Cards, one set of the Studebaker Practice Exercises, and Book I

of one of the leading textbooks on the market The results were uniformly the same Accordingly, the data of Table I, obtained from the textbook, may be considered typical. The averages refer to the average number of times a child was called upon to use each of the combinations Under the caption "Range of Practice" I have listed the smallest and the largest number of times any combination occurs. All of the problems were worked out and a record was made of every combination used. Thus in a problem in long division, a record was made of the division, addition, subtraction, and multiplication involved.

Assumptions. In order to make this count, certain assumptions were necessary They are as follows: (1) It was assumed that all addition is made from the bottom of the column *upward* (2) When carrying is involved, it was assumed that the amount carried was to be added in *at the beginning* of the next column. (3) In multiplication where carrying is involved it was assumed that the amount carried was added as a higher-decade addition For example, in the case of $\begin{array}{r}53\\ \underline{5}\end{array}$ it was assumed that the addition involved is 25 and 1 rather than the reverse. (4) In the case of "borrowing," it was assumed that the one which was borrowed was added to the next figure in the subtrahend (5) In case addition was presented in the equation form, it was assumed that the direction of addition was from *left to right* unless it was necessary to recopy the numbers In the latter case it was assumed that the first number on the left was written at the top of the column (6) The simple division facts which the child uses in estimating the quotient in long division were counted even though the first estimate might be wrong For example, in the exercise $49\overline{)1274}$ a count was given to $12 \div 4$.

TABLE I AMOUNT OF PRACTICE PROVIDED BY BOOK I OF A WELL-KNOWN AND WIDELY USED TEXTBOOK

	AVERAGE PRACTICE	RANGE OF PRACTICE
Addition (App , pp. 155–56)		
First 143 pages	32 5	2 to 174
Entire book	69.4	14 to 335
Subtraction (App., pp 157–59)		
First 143 pages	25 1	2 to 120
Entire book	55.3	8 to 293
Multiplication (App , pp 159–61)		
First 143 pages	12 75	0 to 65
Entire book	68 3	11 to 323
Even division (App., p. 161)		
First 143 pages	14 4	0 to 38
Entire book	27.4	1 to 133
Division with remainders (App , pp 169–72), entire book	3.3	0 to 40
Higher-decade addition to sums of 40 (App , pp. 163–65), entire book	16 6	0 to 64
Higher-decade addition up to sums of 40 with bridging (App , pp 164–65), entire book	2 9	1 to 46
Higher-decade addition involved in multiplication (App , pp. 165–66), entire book	13 3	0 to 64
Higher-decade addition with bridging involved in multiplication (App , p 166), entire book	10 9	0 to 64
Division with zero quotients excluding the zero dividends (App., p 172), entire book	3.7	0 to 15
Division with zero dividends (App , p 161), entire book	27 7	8 to 44
Division involving subtraction with bridging (App , pp 167–69), entire book	1 5	0 to 7

An inspection of the averages and ranges given above raises some interesting practical questions. How many times should a child have used the fundamental com-

binations by the time he is promoted into grade five? By common consent it is assumed that the fundamentals should be so well mastered by that time that the emphasis thereafter may be placed almost entirely on their application in the various situations in which they have proved useful in human experience

Amount of practice needed How much practice should a child have in order to insure this required mastery? This very important question has been almost entirely overlooked by previous writers on the subject. Thorndike,[1] in lieu of anything better, hazards an assumption as to the answer. He says:

> "For one of the easier bonds, most facilitated by other bonds (such as $2 \times 5 = 10$, or $10 - 2 = 8$, or the double bond $7 =$ two 3's and 1 remainder) in the case of the median or average pupil, twelve practices in the week of first learning, supported by twenty-five practices during the two months following, and maintained by thirty practices well spread over the later periods should be enough . . For bonds of ordinary difficulty, with average facilitation from other bonds (such as $11 - 3$, 4×7, or $48 - 8 = 6$) in the case of the median or average pupil, we may estimate twenty practices in the week of first learning, supported by thirty and maintained by fifty practices well spread over the later periods. . . . For bonds of greater difficulty, less facilitated by other bonds (such as $17 - 9$, 8×7, . . .), the practice may be from ten to a hundred per cent more than the above."

Thus the total amount of required practice is estimated at 67, 100, and 110 to 200, according to the inherent difficulty of the combination If this estimate is a reasonable one, it is up to teachers and supervisors to see that this amount of practice material is provided for the 1237 combinations which a child must master[2] If we

[1] Thorndike, *The Psychology of Arithmetic*, p 133

[2] See pages 155 ff

give 150 practices to the 185 hardest combinations and 67 to the remainder of the 1237, the total number of practices would be something less than 100,000 This is somewhere near the number contained in the ordinary three-book series extending through eight grades In other words, the present textbooks contain only about three fourths of what they should contain if Thorndike's estimate is to be met by the end of the sixth grade.

Range of practice A more serious problem presents itself when one looks at the *ranges* of practice as shown in Table I. In addition $\begin{array}{c}9\\ \underline{0}\end{array}$ occurs five times, while $\begin{array}{c}1\\ \underline{1}\end{array}$ occurs 335 times

In subtraction $\begin{array}{c}7\\ \underline{0}\end{array}$ occurs eight times, while $\begin{array}{c}1\\ \underline{1}\end{array}$ occurs 293 times.

In multiplication $\begin{array}{c}9\\ \underline{9}\end{array}$ occurs eleven times, while $\begin{array}{c}2\\ \underline{2}\end{array}$ occurs 323 times.

In division $81 \div 9$ occurs four times, while $4 \div 2$ is found 133 times In division with remainders and in higher decade addition a total of 95 combinations do not occur at all, and the maximum is always far below the minimum requirement, according to Thorndike's estimate To make matters worse, if possible, there is a general tendency to give the least practice to what is inherently the most difficult and the most practice to what is easiest Under circumstances such as these it is certainly out of the question to expect to get better accuracy than we have been getting In fact, it seems remarkable that the children do as well as they do Our teachers are doing well when we consider the crude tools with which they work.

The Need for Supplementary Practice Material

The problem before us is quite clearly that of providing better and more adequate practice material This does not necessarily mean that the textbooks should all be rewritten It may even be impossible to write one that would apportion the amount of drill to the inherent difficulty of the task Furthermore, it is not certain that it would be advisable to use such a book even if it could be written. A textbook of this sort would be very likely to violate one pedagogical principle in seeking to observe another. Difficult combinations must have adequate practice, but this practice should not, as a rule, be given when other difficulties are present When children are being introduced to new topics, such as short division with remainders, long division, or fractions, the new element should be introduced in connection with combinations which are inherently as easy as possible The observance of this principle necessitates a large amount of over drill upon easy combinations, but the benefit derived is worth the cost.

For another reason also, it would be unwise to throw overboard or radically change the textbooks which we now have The estimate given by Thorndike is in terms of averages only. But that which suits the average child is usually extremely out of place for the child who is far above or far below the average.

For both of these reasons it seems best to use the textbooks somewhat as they are and get at the problem through supplementary practice material in card form. Much practice material is now available in this form, but all of it has unfortunately been based upon the theory of transfer. The relative amount of practice is almost

identical with that of the textbooks The same combinations are overdrilled in both and the same important combinations are entirely omitted in both

Both supplementary and textbook practice materials are also open to one further objection Both fail to afford certain essential practice material *when it is needed* This is true both in single combinations and in groups of combinations In one widely used textbook the child meets 0 plus 8 for the first time on page 112 The combination 8 plus 5 occurs only three times in the first 191 pages. In subtraction no drill is provided for the combinations used in short division before short division is introduced. In column addition the child must add small numbers to large ones without any preliminary practice. In carrying in multiplication the conditions are the same. In multiplication, 1 times 1 occurs for the second time on page 183, and 8 times 3 occurs for the third time on page 112. In short division almost no practice is afforded in the use of 1 as a divisor, and the amount of drill is inadequate for cases where a number is to be divided into itself All of these combinations occur frequently in compound multiplication and in long division Some of the difficulty of these latter processes is due to insufficient preliminary practice upon the combinations involved. Trouble with zero in the quotient is notoriously common because little or no practice is afforded where the divisor is greater than the dividend.

What to do in Long Division

Long division is troublesome because too many difficulties are presented at once. A study of the errors which children make in long division shows that the centers of trouble lie (1) in getting acquainted with the new form, (2) in carrying, (3) in borrowing, and (4) in estimating

the quotient including the use of zeros. These difficulties should be introduced one at a time. The worst difficulty is that of estimating the quotient It would seem reasonable, therefore, to postpone the consideration of all problems where the estimated quotient is likely to be wrong until all of the other difficulties are mastered. Some of the textbooks have made an effort to do this, yet one of the most widely used books introduces the child to long division through the problem $15\overline{)240}$ This exercise involves carrying, borrowing, a zero combination and trouble in estimating the quotient in addition to a new form of operation. There is no excuse for such a thing as this. There are 8100 problems in long division which involve two figures in the divisor and two figures in the quotient. Three hundred of these contain no major difficulty other than the new form of operation. One thousand others are equally easy with the exception of the zero combinations which they contain. Twelve hundred contain carrying as the only major difficulty, and more than four hundred involve borrowing as the only major trouble. More than half of the 8100 exercises afford no trouble in estimating the quotient for children who have had a reasonable amount of the right kind of practice in short division. The general rule for teaching long division should be: *Present the examples in such order that one new difficulty and only one is presented at a time No child should be called upon to solve a problem involving all of the difficulties until he has mastered exercises involving each one separately.* Groups of examples arranged according to difficulty are given on pages 173 ff.

The best way to obviate the defects of the present practice material is to provide a set of practice cards which will afford drill upon that which is needed *when* it is needed. These cards should supply drill *directly* upon

those combinations which are missing or of infrequent occurrence in the textbooks With cards of this sort the teacher who makes an effort to diagnose the errors of individual children will be able to correct individual weaknesses by the mere manipulation of the practice cards The cards can be constructed so that each child may know just why he failed to get a correct answer to any particular example. The pupil will be made aware of just how much he knows and just how much remains for him to learn. The cards should stress certain individual combinations which particular pupils miss. Special drill should be afforded in the higher-decade addition combinations listed on pages 163–65, and upon the subtraction involved in short division shown on pages 167–69. Additional drill should also be provided on examples which have zeros in the quotient Lastly, plenty of practice should be afforded in long division in accordance with established principles of teaching relative to order of presentation

Summary

The principles governing practice have not been observed by those who have published our most widely used practice material Combinations which are inherently difficult receive relatively less practice The existence of type errors has been largely overlooked. Our present practice material is unsatisfactory both in amount and character A large number of combinations in higher decade addition and in division with remainders do not occur at all, while many others are practiced so few times that they might as well have been omitted entirely On the other hand, some easy combinations are overlearned beyond all reason. In some instances where a fair amount of practice is provided, the bulk of it comes too late to be

of value Prerequisites occur for the first time along with the things for which they are prerequisite. This is particularly true in multiplication and division

These unfortunate conditions cannot be remedied by textbooks alone. Suitable practice material in loose-leaf or card form is essential, if we are to provide adequately for individual differences and specific needs.

CHAPTER VII

WHAT TO TEACH IN EACH GRADE

The Work for Grade One

In the first grade the main task is to give the pupils a mastery of reading. For this reason it does not seem advisable to require formal work of any sort in arithmetic This does not mean, however, that no number work is to be taught during the first school year. As a matter of fact, the first year is a very important period Children who are learning to read words can easily learn to read numbers also The reading of numbers comes most naturally in connection with the page numbers in the primers and readers. Children at this age like to count, and there are numerous situations in which a need for counting arises

The Need for Objective Experience

One general rule applies to the number work of the first grade It should always be taught in connection with objects No doubt this is the manner in which numbers first came into use by the human race. Rousseau tells us that all of the child's learning should come in that way. Froebel believed that a child's education consisted in his *realization* of or reliving the experience of the race This doctrine breaks down when applied generally In the case of arithmetic, however, there is abundant reason for allowing little children to proceed much as the race did in the discovery of numbers The children are naturally interested in such questions as: How many fingers and toes have you? How many pupils

are in this room? How many pages have we read? How much money have you? When such situations arise, the teacher has a real opportunity to teach the children to count. Most children should be able to count to one hundred before they reach the second grade. Teach also the counting by tens and fives from 10 to 100 and 5 to 100 respectively

In their reading the children will be very likely to meet the numbers in the form "three," "five," etc. On the pages of their books these numbers will appear as "3" and "5" On clocks and watches they often appear in the forms "III" and "V" Teachers should make a special effort to see that children can recognize the smaller numbers in these three forms. The word forms should be learned only as they occur in the child's reading The Roman numerals should be known up to XII as a basis for telling time by the clock The Arabic numerals should be known up to 100.

Number work arises very naturally in connection with common units of measure, such as pints, quarts, inches, feet, yards, pounds, and the several coins in common use. Teachers should give particular attention to the teaching of numbers in cases where the number element is somewhat obscure Little children are naturally perplexed when they are told that one small silver dime is the equivalent of ten copper cents or that a certain bag contains two pounds

The first grade is also the place to begin to provide the experiences essential to the later handling of reasoning problems The meaning of the common units of measure can be developed only through actual experience with foot-rulers, pint cups, nickels, and the like Each child should have the opportunity actually to manipulate these measures himself. The meaning of addition, subtraction,

multiplication, and division may easily be taught by the use of objects If one child has eight marbles and another has five, and there is need to know how many both together have, the natural and right thing to do is to bring both sets of marbles near together and count all of them. Similar procedure applies to the other processes as well. More or less experience of this sort is positively essential and indispensable to the understanding of the words "add," "sum," "subtract," "remainder," and the like. In multiplication the meanings of the words "times" and "product" must be developed in this self-same way. Many teachers prepare large numbers of small pasteboard squares with which the children build larger squares and rectangles. Thus, "three times four" becomes associated with a rectangle built of the pasteboard squares with three squares on one side and four on the other. This is also an excellent method of showing that "three times four" and "four times three" are the same.

The work in division grows naturally out of the work in multiplication A three-by-four rectangle made of pasteboard squares contains rows and columns each of which represents a third and a fourth of twelve respectively. The better children of the grade may even be taught to find three fourths of twelve, two thirds of fifteen, and the like by means of the squares. The work with the squares or other objects is absolutely necessary to the success of the work in the upper grades It should be given, therefore, either at the last of the first year or the first of the second.

The amount of work suggested here will not be sufficient for the brighter children Such children should be encouraged to go on with their counting and reading of numbers. They will be able to dispense with the objects

very quickly. As a general rule no child should be allowed to work with objects any longer than is necessary to form a basis for the more abstract work later.

The Work for Grade Two

The second grade is the place where fixed periods in the daily schedule are first assigned to number work Two questions of very great importance must be answered, however, beforehand: First, unless much valuable time and energy are to be wasted, the teacher must find out whether or not all her pupils have reached sufficient mental maturity to do formal work in arithmetic. Second, she must know whether or not they have had sufficient experience in dealing with objects to constitute a basis for the work which is to follow The mental level of the child is best established by the use of an intelligence test. Where this is impossible, an inspection of the grades made by the child during the previous year will have to be used as a basis of estimate If the child's grades were low and if he fails to learn or retain the easy combinations, the teacher will be justified in the belief that she should wait until he grows a little more before she devotes much extra time to him

Inventory for grade two. To find out whether or not the pupils have had sufficient experience to constitute a basis for advanced work, it is recommended that an informal inventory test be used The following form is suggested. The questions are addressed to each of the children, but the information may generally be obtained from all at once.

An Informal Inventory Test for Pupils Entering the Second Grade

1 Show me how well you can count (for individuals). Count by 10's to 100 Count by 5's to 100.

2 Show me how many numbers you can write (for the group).
3 (To be given orally.) Write for me with figures: 25, 16, 87, 32, 12, 19, 43, etc
4 Make a dozen dots like this (giving a sample)
5 Make a mark one inch long
6 Show me a foot-ruler, a yard stick, a pint cup, a quart cup.
7. How much do you weigh?
8 How many days in a week?
9 How many months in a year?
10 Write and tell me when your birthday comes
11 How old are you?
12 See how many pages there are in your reader
13 Which would you rather have, a penny or a nickel? A dime or a nickel? A dime or a quarter? A quarter or a half-dollar?

Questions such as these will give the teacher an idea of what the children know The prerequisite experience for successful number work in the second grade is not well known, but the following words may be considered as an approximate arithmetical vocabulary which the children should possess when they enter this grade:

all	for	long	pound
as many as	give away	more	price
buy	have left	nickel	right
cent	high	none	sell
day	hour	part	whole
deep	how many	pay	wide
dime	how much	pens	yard
dozen	inch	penny	zero
foot	left	pint	

It is almost certain that very few pupils have all this information upon entering the second grade, but each child will have some of it, and what he does not know he should learn as soon as possible.

It will be well to review during the first few days the work which has been done with the pasteboard squares so as to fix again firmly in the minds of all the children

the meanings of the words add, subtract, multiply, and divide and in order to build up a *sense* of when to use each process.

The formal work of the second grade is as follows:

Notation and enumeration. The reading and writing of numbers to 1,000.

Combinations. The following are to be learned.

ADDITION

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	0
2	2	2	2	2	2	2	2	
1	2	3	4	5	6	7	0	
3	3	3	3	3	3	3		
1	2	3	4	5	6	0		
4	4	4	4	4	4			
1	2	3	4	5	0			
5	5	5	5	5				
1	2	3	4	0				
6	6	6	6					
1	2	3	0					
7	7	7						
1	2	0						
8	8							
1	0							
9								
0								

Subtraction

1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1
	2	3	4	5	6	7	8	9
	2	2	2	2	2	2	2	2
		3	4	5	6	7	8	9
		3	3	3	3	3	3	3
			4	5	6	7	8	9
			4	4	4	4	4	4
				5	6	7	8	9
				5	5	5	5	5
					6	7	8	9
					6	6	6	6
						7	8	9
						7	7	7
							8	9
							8	8
								9
								9

0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0

Multiplication

1	2	3	4	5	6	7	8	9	0
1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	0
2	2	2	2	2	2	2	2	2	2
1	2	3	4	5	6	7	8	0	
3	3	3	3	3	3	3	3	3	
1	2	3	4	5	6	0			
4	4	4	4	4	4	4			
1	2	3	4	5	0				
5	5	5	5	5	5				
1	2	3	4	0					
6	6	6	6	6					
1	2	3	0						
7	7	7	7						
1	2	3	0						
8	8	8	8						
1	2	0							
9	9	9							

Division

The division combinations will correspond exactly to those in multiplication.

Higher-Decade Addition

10 and 0	11 and 0	12 and 0	13 and 0	14 and 0
10 and 1	11 and 1	12 and 1	13 and 1	14 and 1
10 and 2	11 and 2	12 and 2	13 and 2	14 and 2
10 and 3	11 and 3	12 and 3	13 and 3	14 and 3
10 and 4	11 and 4	12 and 4	13 and 4	14 and 4
10 and 5	11 and 5	12 and 5	13 and 5	14 and 5
10 and 6	11 and 6	12 and 6	13 and 6	
10 and 7	11 and 7	12 and 7		
10 and 8	11 and 8			
10 and 9				

15 and 0	16 and 0	17 and 0	18 and 0	19 and 0
15 and 1	16 and 1	17 and 1	18 and 1	
15 and 2	16 and 2	17 and 2		
15 and 3	16 and 3			
15 and 4				

20 and 0, etc , to 20 and 9	30 and 0, etc , to 30 and 9
21 and 0, etc , to 21 and 8	31 and 0, etc , to 31 and 8
22 and 0, etc., to 22 and 7	32 and 0, etc , to 32 and 7
23 and 0, etc., to 23 and 6	33 and 0, etc., to 33 and 6
24 and 0, etc , to 24 and 5	34 and 0, etc , to 34 and 5
25 and 0, etc., to 25 and 4	35 and 0, etc , to 35 and 4
26 and 0, etc , to 26 and 3	36 and 0, etc., to 36 and 3
27 and 0, etc , to 27 and 2	37 and 0, etc., to 37 and 2
28 and 0, 28 and 1, 29 and 0	38 and 0, 38 and 1, 39 and 0

The *groups* of combinations listed above should be taught in the order in which they are given Within the groups, however, the combinations should be taught in chance order *Do not teach them in the order given here.* It is advisable to teach the addition and subtraction combinations together. The same is also true for multiplication and division The combinations should be taught both in the form given and in the equation form Children who know $\begin{array}{r}3\\+\,2\\\hline\end{array}$ and $\begin{array}{r}2\\\times\,4\\\hline\end{array}$ are often perplexed when they meet with $2 + 3$ and 4×2. The equation

form is of great practical value both in life and as a preparation for algebra.

The higher-decade addition combinations should be taught only in the form in which they are listed The order should be varied There is no objection to the use of exercises in the form $\begin{array}{r}4\\ \underline{25}\end{array}$ or $\begin{array}{r}13\\ \underline{6}\end{array}$, but this sort of practice is of no value in preparing children to add columns involving carrying.

Application of the combinations. The combinations listed above should be practiced in as many different settings as possible Column addition should be begun in the second grade, but must not involve combinations other than the ones listed here. Many exercises such as the following should be used.

	2		22	63	12	13
2	3	4	3	2	21	2
3	1	3	1	3	31	31
4	2	11	2	31	2	23

Much informal practice should be afforded in one-step reasoning problems, but these should under no circumstances require the use of combinations other than those listed above. If any of the children show a tendency to juggle the figures, they should be provided immediately with further experiences with pasteboard squares and other objects The objects mentioned in the reasoning problems should never be those with which the children have had no experience. The problem, "If one sibtad costs 5 cents, how much will four sibtads cost?" will prove at least slightly embarrassing to many adults One should think of four times five, but there is an almost uncontrollable tendency to wonder what a sibtad is. This is an irrelevant thought and is absolutely valueless for the problem in

hand. It leads in the direction of confusion and error. Such irrelevant ideas are potent factors in perturbing the little reasoner. Therefore, avoid all problems which involve situations or names which are strange to the pupils.

Vocabulary. The children should all know the meaning of every word in every problem before they are asked to solve it. This means that every such word must be in the child's reading vocabulary. At least one fifth of the trouble which children have with reasoning problems is due to inability to read the problem The author knows this to be absolutely true because of his study of thirty thousand errors.[1] The children will continue to need help from time to time with the words given in the inventory list on page 76. Before the end of the year the teacher should also be sure that the children know the meaning of each of the following words:

add	dollar	hour	none	right
addition	difference	how many	opposite	sell
all	distance	how much	part of	separate
amount	divide	inch	pay back	share
as many as	dividend	increase	penny	square
bank	division	left	pens	subtract
bushel	divisor	less	pint	subtraction
buy	dozen	long	pound	take from
cent	due	minute	price	the least
change	equals	missing	product	times as many
charge	foot	more	purchase	times
column	for	multiplicand	quarter	triangle
cost	gill	multiplication	quarterly	underneath
cube	give away	multiplier	quotient	whole
day	half dollar	multiply	rectangle	width
deep	have left	minus	remainder	yard
depth	height	nickel	remove	zero
dime	high			

The following abbreviations and signs should be known: doz., in., ft., yd., bu., pk., qt., pt., lb., ¢, $, +, −, ×, and ÷.

[1] See page 38.

Teach also one half of every even number through 20 and one third of every multiple of 3 through 24.

THE WORK FOR GRADE THREE

Inventory Wise teachers will always begin the year with an inventory of what the children know concerning what has gone before. Ask the pupils to write the numbers from 1 to 45 in a column, leaving space at the right of the column Read one of the combinations listed on page 77. Say, for example, "How much is 2 and 3? Write what you think this is just after the number 1 on your tablet." See that all get it in the right space Then say, "I am going to read some more combinations to you. Please write the answers after 2, 3, 4, etc. I am going to read them quite rapidly You will have to hurry if you keep up. If you do not know the right answer, skip the space and go to the next number." Read the remaining combinations *in chance order*. Do not use the order given on page 77.

In order to help some children to get back on the track after they have been "lost," it is well to say, "Number 10 is so and so." Then proceed as before. Both teacher and pupil will need preliminary practice before the final try-out is made The pupil will need practice to enable him to skip correctly when he does not know the combination The teacher will need practice to enable her to read the combinations regularly and at the correct rate of speed. A reasonable time between readings for the beginning of grade three is six seconds Use a similar procedure for the combinations given on pages 77, 78, and 79. List all combinations missed and remember that trouble with the preceding combination often causes a child to miss a combination which he would know otherwise.

In this way each child will learn exactly what he has

forgotten or failed to master in the previous year's work It is worth while also to test the children on the types of column addition and reasoning problems which they studied in the second grade A check should likewise be taken of the mastery which the children have of the special vocabularies and abbreviations listed on pages 76 and 82 This inventory will afford a basis upon which teachers can intelligently plan their review work.

Notation and enumeration The reading and writing of numbers to 10,000 should be taught

Combinations *Teach first the combinations which were missed on the inventory.* Then take up those in the following lists:

ADDITION

1	2	3	4	5	6	7	8	9
9	9	9	9	9	9	9	9	9
2	3	4	5	6	7	8	9	
8	8	8	8	8	8	8	8	
3	4	5	6	7	8	9		
7	7	7	7	7	7	7		
4	5	6	7	8	9			
6	6	6	6	6	6			
5	6	7	8	9				
5	5	5	5	5				
6	7	8	9					
4	4	4	4					
7	8	9						
3	3	3						
8	9							
2	2							
9								
1								

Subtraction

10	10	10	10	10	10	10	10	10	10[1]
1	2	3	4	5	6	7	8	9	10

11	11	11	11	11	11	11	11	11[1]
2	3	4	5	6	7	8	9	10

12	12	12	12	12	12	12	12[1]
3	4	5	6	7	8	9	10

13	13	13	13	13	13	13[1]
4	5	6	7	8	9	10

14	14	14	14	14	14[1]
5	6	7	8	9	10

15	15	15	15	15[1]
6	7	8	9	10

16	16	16	16[1]
7	8	9	10

17	17	17[1]
8	9	10

18	18[1]
9	10

19[1]
10

Multiplication

9	7	8	9	6	7	8	9	5	6	7	8	9
3	4	4	4	5	5	5	5	6	6	6	6	6

4	5	6	7	8	9	4	5	6	7	8	9	3
7	7	7	7	7	7	8	8	8	8	8	8	9

4	5	6	7	8	9
9	9	9	9	9	9

[1] To be used if the pupils are taught to "pay back" after borrowing by adding one to the subtrahend. Otherwise they may be omitted.

Division

The division combinations correspond directly to the multiplication combinations listed above

Higher-Decade Addition

11 and 9	16 and 4	18 and 4	21 and 9
12 and 8	16 and 5	18 and 5	22 and 8
12 and 9	16 and 6	18 and 6	22 and 9
13 and 7	16 and 7	18 and 7	23 and 7, etc to 23 and 9
13 and 8	16 and 8	18 and 8	24 and 6, etc to 24 and 9
13 and 9	16 and 9	18 and 9	25 and 5, etc to 25 and 9
14 and 6	17 and 3	19 and 1	26 and 4, etc to 26 and 9
14 and 7	17 and 4	19 and 2	27 and 3, etc to 27 and 9
14 and 8	17 and 5	19 and 3	28 and 2, etc to 28 and 9
14 and 9	17 and 6	19 and 4	29 and 1, etc to 29 and 9
15 and 5	17 and 7	19 and 5	
15 and 6	17 and 8	19 and 6	
15 and 7	17 and 9	19 and 7	
15 and 8	18 and 2	19 and 8	
15 and 9	18 and 3	19 and 9	

Fractions

Review 1/2 and 1/3 Teach 1/4, 1/5, 1/6, 1/7, 1/8, and 1/9

The combinations to be taught in higher-decade addition as a prerequisite for multiplication are given in the Appendix, pages 165–66. Two groups are given composed of those which are relatively easy and those which are relatively difficult The easy ones should be taught first. No multiplication involving carrying is to be presented until the corresponding addition combinations have been mastered. The combinations in simple addition, subtraction, and multiplication are to be taught in the equation form also The higher-decade additions are to be taught *only* in the equation form. The children should have by this time a complete mastery of the signs +, −, ×, ÷, and =. The plus and equality signs

should be used with the higher-decade additions along with the words " and " and " equals "

Application of the combinations. An application of any of the combinations listed on pp 84–86 invariably involves " borrowing " or " carrying " Both of these terms are often included under the one term " bridging " The mastery of these two difficulties is one of the main purposes in the third grade Further suggestions on these troublesome points are given on page 126. After the children have learned the combinations listed above, they are prepared to do all sorts of column addition, *provided not more than three is carried in any case.* Simple and compound multiplication should be taken up Short division may be introduced informally, but only the simplest forms should be used.

The third grade is the place to begin formal work with one-step reasoning problems No problems with more than one step should be used Particular effort must be made to see that a sufficient amount of practice is given to establish a sense of which of the four processes is to be used One-step problems involve either addition, subtraction, multiplication, or division No evidence is at hand to show how much practice is required to yield a mastery of each type It is probable, however, that the types are not of equal difficulty From data at hand it seems safe to estimate that the order, ranging from easy to difficult, is addition, multiplication, subtraction, and division The teacher should note particularly any pupils who fail to respond properly to any of these types and should provide additional practice with the paste-board squares to remedy the particular type of weakness which the pupil exhibits[1] Two fundamental principles should always be observed

[1] See page 74.

(1) Never allow a pupil to attempt a problem with things he has not experienced

(2) Do not present problems which have no practical application in real life.

An observance of the latter principle involves a careful scrutiny of each problem, since problems which violate it are to be found even in our best textbooks.[1]

The Work of Grade Four

Inventory. The inventory to be given at the beginning of the fourth grade includes for the most part the combinations given on pages 77–80 and 84–86. The directions showing just how to proceed are given or suggested on page 83. The time limits are five seconds for the beginning of Grade IV.

Exercises in (1) column addition involving carrying, (2) subtraction with borrowing, and (3) multiplication with carrying are also necessary. The following are samples:

$$\begin{array}{r} 5860 \\ 689 \\ 97 \\ 468 \\ \underline{9875} \end{array} \qquad \begin{array}{r} 50040 \\ \underline{6976} \end{array} \qquad \begin{array}{r} 8976 \\ \underline{7} \end{array}$$

When a child makes a mistake in any of the preceding types, *be sure to have him do his work aloud so that you may discover the exact nature of his error.*

In the matter of reasoning problems the aim is to make the pupils sensitive to the process to be used in each type of one-step problem. The following are samples of prob-

[1] For further discussion of this very important point see Thorndike, E. L., *Psychology of Arithmetic*, pages 92 ff. New York, The Macmillan Company, 1922.

lems which each child should be able to solve when he enters the fourth grade

1 I went to the store for mother and bought one dozen eggs for 45 cents, a basket of grapes for 40 cents, a pound of butter for 60 cents, and a pound of crackers for 20 cents. How much money did I spend?

2. I bought one dozen rolls for 15 cents I gave the clerk fifty cents How much change did I receive?

3. If one dozen oranges cost 58 cents, how much will seven dozen cost?

4 We bought a desk for $40 00. Each month we pay $5 00 How long will it take to pay for the desk?

The children should know all of the abbreviations mentioned on page 82, all of the signs of operation, and the equality sign

The first task will be to teach each pupil the things he has missed in the inventory.

Notation and enumeration The children should learn to write and read all Arabic numbers up to 1,000,000 and all of the Roman numbers in common usage.

Fractions Review the fractions taught in grade three Teach 1/10, 2/3, 3/4, 2/5, 3/5, 4/5, and 5/6

Combinations The main aim on the formal side during the fourth grade will be the mastery of long and short division. The work must begin with a study of the subtraction involved in short division This subtraction is not of the written sort and has received no practice up to this time Be sure, therefore, to see that your pupils have a mastery of every subtraction combination listed on pages 167–69. These exercises should not be used in the column form — at least such usage is of little or no value as a prerequisite for division. Teach these combinations either in the form given or in some form in which the child is required to deal mentally with the numbers *as wholes*.

Short division without carrying has been studied in the third grade to some extent, but it will be necessary to start at the beginning again since the third-grade work was entirely informal in character.

The order of procedure is as follows: Begin with exercises which do not involve carrying Only a few of these are in existence in simple form Samples are as follows:

$2\overline{)22}$	$2\overline{)20}$	$2\overline{)24}$	$2\overline{)42}$	$2\overline{)60}$	$2\overline{)40}$
$2\overline{)80}$	$2\overline{)26}$	$2\overline{)28}$	$2\overline{)44}$	$2\overline{)46}$	$2\overline{)48}$
$3\overline{)30}$	$3\overline{)33}$	$3\overline{)39}$	$3\overline{)60}$	$3\overline{)63}$	$3\overline{)36}$
$3\overline{)66}$	$3\overline{)69}$	$3\overline{)90}$	$3\overline{)93}$	$3\overline{)96}$	$3\overline{)99}$
$4\overline{)40}$	$4\overline{)44}$	$4\overline{)48}$	$4\overline{)80}$	$4\overline{)84}$	$4\overline{)88}$

The list can be extended by using three or more figures in the dividend, but in each case *every additional digit used must exactly contain the divisor* Short division should always be introduced by means of these exercises because the only new element is the presence of more than one figure in the quotient.

When this type of division is well established, review the easier portion of the subtraction on pages 167–68. Then proceed to the short-division exercises which *begin* with the *easy* division combinations listed on pages 169–71, and end in such a manner as to leave no remainder when the division is complete. Carrying will be the new element here, and special attention must be paid to its mastery The list on pages 169–71 can be used as an inventory at any time, as described on page 83 As soon as any child answers 95 per cent correctly, review him on the more difficult subtraction list (page 168) and then let him take up the more difficult combinations on pages 171–72, always adding a digit to the dividend so that no remainders

will be obtained Use the hard lists in subtraction and division as inventories from time to time in order to discover pupils who are ready to proceed to the next step.

The next step will be to familiarize the children with exercises in which there is a remainder. It is suggested that the pupils be taught for the present to write their quotients as so much with so much over. The first problems involving remainders should be those made up of the easier combinations in both subtraction and division.

The final step in short division will be the mastery of situations in which the child meets dividends which are less than the divisors. Special attention should be devoted to these exercises because they fortify the pupil against the notorious error of failing to write zeros in the quotient when necessary. This type of drill is also an indispensable prerequisite for percentage Guard the child against the idea that the dividend is always greater than the divisor. The list of exercises on pages 171–72 contains all of the forty-five simple elements of this sort. Use it frequently as an inventory.

When the child has progressed thus far, he is ready for long division This part of our number work is of well-known difficulty. Most of the trouble is the result either of presenting too many new difficulties at once or of presenting them in the wrong order. It is quite essential, therefore, that the teacher follow the order suggested on pages 173 ff in the presentation of the exercises. The easiest divisors in long division are 10, 11, 20, 21, 30, 31, etc. With them the child can devote most of his attention to the single new element involved, namely, the form of operation. Such exercises are listed as Class A on page 173.

The classes of exercises which follow in the appendix are planned so as to introduce one new major difficulty and only one at a time. Class B includes exercises involving

carrying; Class C, those involving borrowing; and Class D, those involving both carrying and borrowing In Class E the pupil meets for the first time situations in which the estimated quotient will ordinarily be wrong The exercises in this and the following classes should *under no circumstances* be presented before classes A, B, C, and D are mastered. It is probably advisable to give drill in examples having more than two figures in the quotient and divisor before confronting the child with exercises which give special trouble in the estimation of the quotient. The brighter children will be able to master long division during the fourth grade, but it is quite probable that many pupils should not even attempt exercises as difficult as those in Class G. The division ladder given on page 178 will be of value in ascertaining how far each pupil has progressed, provided, of course, that there has been no practice upon the test before it is used

Reasoning problems. The work in problems should be continued much as in the third grade Few two-step problems are advisable in grade four One sort should be taught, however Pupils should be able by the end of the year to tell how much change should be received after several articles have been purchased Three-step problems of this sort should also be taught because of their great social importance. The processes involved are, of course, multiplication, addition, and subtraction Valuable practice on one-step problems may be obtained in the fourth grade by asking the pupils to make up one-step reasoning problems in each of the four processes

Applications of the combinations Continued practice should be afforded in all of the work outlined up to this point in all sorts of settings; but the numbers should not be larger than those in ordinary use outside of school. Some drill should also be given in the addition of decades

higher than those suggested heretofore. It is hoped, however, that sufficient transfer will occur so that the remaining higher decade additions may be mastered without any great amount of practice The teacher should conscientiously persist in requiring pupils to do their work aloud when mistakes have occurred that cannot be detected accurately from inspection This means individual attention to certain pupils, but it is worth the cost since no other method is available for the discovery of serious individual disabilities

Upper Grade Work

A detailed discussion of the work in the upper grades is beyond the scope of this book A few things may be suggested which are known to be essential to those who would use corrective types of arithmetic.

(1) At the beginning of each school year an inventory should be taken to detect any elements from the previous grades which have been forgotten or which were never learned The most important group of errors in each of the upper grades centers around the fundamental combinations which should have been learned in the primary grades The exercises in the appendix of this volume should be given each year to all pupils who show inaccuracy in their work Rugg and Clark [1] found that one of the few major errors in the algebra work of some fifty children was the inability to do accurately the primary number work involved in the algebra exercises. The most sensible thing to do for pupils of this sort is to take an inventory of their arithmetic knowledge and see just what is causing the failure of the pupils in their algebra work

[1] Rugg, H O, and Clark, J R, *Scientific Method in the Reconstruction of Ninth-Grade Mathematics*, pages 180 ff Chicago University of Chicago Press, 1918 (Supplementary Monographs, vol 2, no 1, whole no 7)

(2) The fifth grade is usually the place where the pupils are supposed to obtain a reasonable mastery of common fractions The facts of corrective arithmetic which apply in this grade are not well worked out. The following, however, are worthy of consideration.

Confine your efforts to the teaching of those fractions which are of most social value. Most of the fractions considered should have denominators less than 13

Guard against mere juggling with denominators and numerators Two possibilities are at hand to meet this difficulty. One is the out-and-out memorization of the most important combinations of fractions Pupils should know at sight the answer to $\frac{1}{2} + \frac{1}{3}$ just as they know 7 times 9. The other is the habit of falling back upon the pasteboard squares as a means of checking answers. Suppose, for example, that a child has said that $\frac{1}{2} + \frac{1}{3}$ is $\frac{1}{6}$. (Many do say so.) Have the child build a rectangle of say twelve squares and let him count out what he gets when he takes out $\frac{1}{2}$ and then $\frac{1}{3}$ of the twelve. This, by the way, is an excellent preparation for the very troublesome exercises on the type "What part of 12 is 10?" and the like.

In grade six the children will be required to gain a mastery of decimal fractions. As a corrective measure in reducing common fractions to decimals, stress again the fact that the divisor is often less than the dividend A good way to do this is through the use of ordinary problems involving money. For example, "Five boys are planning to pull the weeds from a man's garden The man will pay them three dollars. How much will each boy get if they all share equally?"

Experience shows that children have much trouble with the use of the decimal point See to it that the point is placed correctly. In the addition and subtraction of

decimals it will be difficult for the children to keep the points even. The reason for this is the notion, well established by this time, that the right-hand ends of the numbers to be added or subtracted must be kept even. In subtraction, matters are still further complicated by the idea that the minuend always appears larger than the subtrahend In such an exercise as $10 - 3.75$, there is consequently a strong tendency to write the exercise in the form $\begin{array}{r} 375 \\ \underline{10} \end{array}$ and thus to obtain an answer of 365. Further suggestions of spots in which corrective work is needed are given on pages 47, 50, 52, and 54 Each of the types of errors listed on those pages indicates danger spots in the child's path.

Do not fail to give continued practice on the equation form. If your pupils are not well acquainted with this form by the time they reach the upper grades, you cannot do better corrective work than to emphasize it Samples of the exercises follow:

$5 + 1 = ?$	$\frac{1}{2} + ? = \frac{5}{6}$	$.4 \times 2 = ?$
$3 \times 7 = ?$	$2 \div ? = 4$	$3 \times ? = 03$
$? + 6 = 15$	$4 \times ? = \frac{1}{2}$	
$17 - ? = 8$	$\frac{?}{3} = \frac{3}{6}$	
$53 - 48 = ?$		

Remedial Work in the Solution of Problems

Useful and essential corrective work for reasoning problems is given in the samples below. A mastery of this type of material will remove at least sixty per cent of the errors in arithmetical reasoning.

Some things to do for children who fail to understand quantitative relations. Make a list of the important quantitative relations which occur in the problems which you assign. The following list is suggestive:

(1) The relation between cost, expenditure, income, and selling price.

A certain house rents for \$960 a year; the taxes and upkeep cost \$300 a year. How much can I afford to pay for this house in order to clear 6% a year on my investment?

Suppose I have bought the house at this price and am offered \$12,000 for it Should I sell? Why?

(2) The relation between rate, time, and distance.

The distance from New York to Chicago is 980 miles. How long will it take a train to run this distance at the rate of 49 miles per hour?

(3) The relation between length, breadth, and thickness

I need a bin that will contain 240 cubic feet. If I make it 8 feet long and 6 feet wide, how deep must it be?

(4) Relations involving proportion

It takes 9 men 5 days to build a piece of road. How many men would it take to build this road in 3 days?

The following equations relate to the four problems given above. Only the first letters of the words are used

(1) I (income) $-$ E (expenditure) $=$ C (amount saved).
C (cost) $+$ G (gain) $=$ S (selling price).

(2) D (distance) $=$ R (rate) $\times$ T (time).

(3) C (contents) $=$ L (length) $\times$ B (breadth) $\times$ D (depth).

(4) $9 \times 5 = ? \times 3$

In case G is wanted, as in Number 1, start with $C + G = S$. Then teach that C, which is not wanted on the

same side with G, can be put over with S by subtracting C from both sides of the equality sign, as in $G = S - C$

In case D is wanted, as in Number 3, start with $C = L \times B \times D$ Then teach that D can be made to stand alone by *dividing* both sides of the equality mark by L and B, thus getting $C \div L \div B = D$

Practice like the foregoing should be abundant A failure to provide such exercises will insure failure in reasoning problems in both arithmetic and algebra.

Finally do not fail to give frequent corrective work in arithmetic reading. Samples of the type of practice needed for corrective work follow.

Samples of silent-reading exercises prerequisite to the solution of problems. To the pupil · Can you understand your arithmetic problems when you read them? If so, you can do these exercises correctly. After each problem is a list of things which might be done. You are to show that you understand by drawing a line under the word or words which tell the *right thing to do*

(1) Mary wishes to save 35 cents If she has already saved 24 cents, how much more must she save?

Add Subtract Multiply Divide

(2) How many tablets can I buy for 40 cents, if each tablet costs 8 cents?

Add Subtract Multiply Divide

(3) Fred's mother has 3 rows of fruit jars with 9 jars in each row. How many fruit jars has she?

Add Subtract Multiply Divide

(4) Frank had 2 apples and his mother gave him 3 more. How many apples did he then have?

Add Subtract Multiply Divide

Fill in the blanks and underline the words which tell the right things to do. Here you are not to fill all of the blanks.

(5) How much cheaper is it to rent a room for a year at \$12 50 a month than it is to rent another room for the same length of time at \$3.50 a week?

Divide —— by —— Multiply —— by ——
Divide —— by —— Multiply —— by ——

Then Add Subtract Multiply Divide

(6) A farmer sold 3,375 bushels of potatoes at \$.90 a bushel, and wishes to buy land at \$75 an acre. How many acres can he buy?

Divide —— by ——
Multiply —— by ——
Multiply —— by ——

Then Add Subtract Multiply Divide

(7) If a man receives \$7 20 for 8 hours of work, how much should he get for $6\frac{1}{2}$ hours of work?

Add —— to —— Multiply —— by ——
Subtract —— from —— Divide —— by ——

Then Add Subtract Multiply Divide

Some things to do for children who give only partial responses to the problem Most of the trouble here is due to a failure to read the problem completely Exercises like the following will prove helpful.

(1) A boy had 240 marbles and lost $\frac{1}{4}$ of them. How many had he left?

Are you asked to find how many marbles the boy lost?

(2) John sold his rabbit for 20 cents which was $\frac{4}{5}$ of what the rabbit cost him. How much did he pay for the rabbit?

Underline the words which make the following statement true:

John sold his rabbit for (more than, less than) it cost him

Are you asked to find out how much money John lost on his rabbit?

Underline the word which tells the right things to do.

Add Subtract Multiply Divide

CHAPTER VIII

PRINCIPLES OF METHOD

THE fundamental thesis of corrective arithmetic is the idea that the function of the teacher is to help the child to learn things of value in arithmetic which he is capable of learning, but has not learned The problem of every teacher consists, therefore, in finding out what the pupils of a given grade need to learn and can learn and how they may best learn it The teacher's method must consequently be based upon an observance of the laws and principles of growth and of learning. It will be the main purpose of this chapter to indicate the most important of these laws and principles and to show how they apply to the teaching of arithmetic.

THE LAWS AND PRINCIPLES OF GROWTH

Growth means vastly more than mere increase in size. Children are not little men and little women even in a physical sense Until this fact was recognized, there was no sound basis for method in education — in fact, little method was necessary. If children were like adults in every respect save that of size, and if all adults were alike, then any adult could teach equally as well as any other adult He could teach all that he had ever learned by the same method as that by which he himself had learned it. This fallacy more than anything else is responsible for the relatively retarded condition of the science of education Many highly intelligent people even now insist that there is no such thing as a science of education A gradually increasing recognition of the difference between

children and adults is clearing the way, however, for the advancement of educational practice and the development of teaching into an expert occupation.

Differences between children and adults First of all, there are the very apparent differences in size and strength. It is also quite evident that little children cannot coordinate their muscles as well as grown-ups Children are more active physically than adults and are more inclined to play. The interests of children are unique and characteristic of different ages in child life Children are less capable of sustained attention, less able to resist distractions They are inexperienced and therefore incapable of judging or reasoning as well as adults. The attention span of children is relatively limited They are also relatively incapable of thinking or remembering in a logical manner

The teaching of arithmetic is still suffering because of delusion in regard to these matters. The writers of some textbooks and the authors of some standard tests still expect children to deal with huge numbers and to add extremely high columns. These authors are interested in such things and assume that children are likewise interested in them and that they are capable of doing the work involved Some teachers assume that their pupils are interested in knowing why a fractional divisor is inverted or why you square the radius of a circle and multiply it by 3 1416 in order to find the area Some textbooks are still teeming with reasoning problems which no sane person would ever use in a real situation. Indeed, we are barely free from the time when children were compelled to learn "apothecary's weight," "Vermont rule for partial payments," "true discount," and numerous other things solely for the reason that teachers and textbook makers were interested in them. Those who expect to be good

teachers must learn first of all that it is their function to help little children to grow into good adults.

Individual differences A second principle of fundamental importance is the fact that different children grow at different rates. A century ago this fact was totally unrecognized. As a result we have, among other things, the grade system of school organization. If all children grow at equal rates, it follows that all will reach a given stage of development at the same time Hence we expect all six-year-olds to be in the first grade, all seven-year-olds to be in the second, and so on. But the truth of the matter is that some children belong in the first grade at the age of four, while some cannot do first-grade work at the age of twelve. Thus in many schools teachers are trying to teach pupils who are utterly incapable of learning what they are trying to teach, while other pupils in the same grade are bored almost to distraction with tasks which to them are infantile

The application of this principle to the teaching of arithmetic has already been pointed out several times Some children in the second grade should not be expected to do abstract number work and some in grade eight are not able to do arithmetic reasoning. Do not attempt the impossible in your teaching Do not attempt to force stupid little children to overstrain themselves in an effort to learn that which is beyond them.

Varying rates of growth in the same individual The idea that individual children have varying rates of growth both physically and mentally was first set forth in a prominent manner by G Stanley Hall about the beginning of the present century[1] Hall's writings exerted a far-reaching influence and tinged the educational thinking of that whole period The coming of more accurate means

[1] Hall, G. Stanley, *Adolescence*, Preface, page xiii and elsewhere 1904

of measurement, however, has shown that some of Hall's conclusions were overdrawn and overemphasized In recent years, Baldwin and others have been attacking the problem of growth in a more scientific manner.[1] It is now known with certainty that the growth of a normal child is quite regular from year to year both physically and mentally There are variations, but they are not of the spectacular sort which Hall described.

One must not forget, however, that Baldwin's results are for average or normal children only No evidence has been produced to show that the rate of growth is constant for every child Common observation tells us that a long illness often retards physical growth We know too that mental growth is subject to similar variations in individual cases Retardation in the rate of growth is called *arrested development*.

Such a period of arrest may be either temporary or permanent For this reason it is not safe as yet to predict concerning the future growth of any child either physically or mentally We know that the great majority of children who are tall at birth will be tall when grown We know, too, that by far the most of the children who have a mentality on the Binet Scale of eighty per cent of the average for their age will continue to have about eighty per cent mentality on that scale all of their lives But there is a minority who are suffering at any given time from a temporary arrest The teacher should be constantly on the lookout for these few pupils. Sometimes the cause of the trouble is merely poor eyesight, adenoids, or the malfunctioning of some gland If the trouble can be found and removed, the child will often grow with sufficient rapidity to make up for lost time

[1] Baldwin, Bird T, "A Measuring Scale for Growth and Physiological Age," *Fifteenth Yearbook, National Society for the Study of Education*, part 1, pages 11–22.

The application of these facts to the teaching of arithmetic is evident. Children who are suffering from physical defects often fail to keep up with their grades. Sometimes the cause of the arrest in arithmetic is partly emotional. Mary may get low grades in her arithmetic work because she has not had sufficient experience to enable her to understand the reasoning problems or because she responds wrongly to a few of the essential number combinations listed in the Appendix of this book. At the same time she may be doing excellent work in geography, spelling, and reading. Mary's parents soon find that her grades are low in arithmetic, but high in other things. Now, instead of seeking for the cause of the trouble in the arithmetic, both teachers and parents will sometimes cooperate to build up in Mary's mind the idea that she "just can't get arithmetic, but is good in everything else." Such a condition leads straight toward what the psychologists call an inferiority complex. Mary gradually comes to have a permanent feeling of inferiority with reference to arithmetic. Her development, already obstructed by other causes, becomes more and more retarded because of the emotional element involved and finally settles down to a condition of permanent arrest.

Teachers should *guard most carefully* against making prophecies concerning the future progress of a retarded child. Strengthen the pupil's morale all you can. Seek diligently for the cause of the trouble. Do not waste your time and overburden a pupil by attempting to teach him what he cannot learn in his present condition. Provide for him the most stimulating environment possible and *wait for him to grow.*

The Laws of Learning

The law of use. We have seen how mistakes have been made by ignorance or disregard of the laws and principles of growth. It is equally disastrous to neglect or ignore the laws of learning We all have an unfortunate tendency to overemphasize that which is most open to the observation and to ignore that which is somewhat obscure The statement that "practice makes perfect" is a proverb whose origin is lost in remote antiquity. Every man and woman in the street has heard the proverb and believes it Belief is tremendously strengthened by the idea that all men are equal intellectually and emotionally. But the proverb is often not true, and the idea of intellectual and emotional equality is entirely false. When is it that practice *does* make perfect? Every teacher should know this and know it definitely because it is one of the phases of expertness which teachers must have if they are to be experts at all To say that practice always makes perfect is just as absurd as to say that a body in motion will remain in motion forever Both statements are true so long as no opposing forces or obstacles get in the way. In the learning process the opposing forces are produced by the violation of some of the other laws of learning.

The law of effect. In educational psychology the idea that practice makes perfect is called "the law of use." For generations it was thought that all of human learning was subject to this one law. Now we know that just as a body in motion is always subject to the force of gravity, so the law of use can never operate independently of another force. The law which expresses this other force is called "the law of effect." The law of use tells us that things which are practiced or repeated tend to become permanent in the mind. The law of effect says that things which be-

come permanent in the mind *must be satisfying to us.* Practice with things that are annoying does not lead to improvement or permanence Practice in eating breakfast food may make us want breakfast food every morning, but practice in having one's teeth pulled does not make us want to repeat the experience Persons who are greatly afraid of water do not learn to swim Girls who detest sweeping do not make good housekeepers. Boys who hate plowing do not make good farmers. Children who hate school do not learn much Practice under such conditions tends to make us imperfect rather than perfect. Practice makes perfect *only when the results of the practice are satisfying to the person who is practicing*

An old fallacy We are just emerging from a period in which the law of effect was entirely neglected and even opposed. Many of our ancestors believed thoroughly that man was "born in iniquity and conceived in sin" From this it followed quite naturally that children were sinful creatures Therefore, whatever satisfied them was evil and useless, while whatever annoyed them was good and worth while According to such philosophy the function of the teacher was to the teach the child to do "what he did not like to do" A school in which the children were contented and happy was therefore a poor school; and one in which the children were bored and unhappy was a good one.

This philosophy is still in vogue among many laymen Witness the frequent appearance of newspaper cartoons showing how children hate the school. Such a point of view could scarcely be worse for the progress of the science of education The writer remembers an institute which he attended some years ago, in which a young teacher got up and asked timidly how to make a fourth-grade lesson in nature study interesting. There was a prolonged

silence Then the most experienced superintendent in the group arose and said, "The school isn't supposed to be as interesting as a circus The school is a place for work." It would have been far better if her school board had kept the young teacher at home. With such a heritage back of us and with such misconceptions still prevalent among our patrons, it behooves us as teachers to study very carefully the applications of the law of effect and use it wisely.

Avoidance of extremes There are two dangers to be avoided. One is the complete failure to observe the law and the other (which is just as bad) the elevation of the law of effect into an object of worship. Some teachers have grown so enthusiastic that they refuse to have the children practice upon anything that is not at first sight interesting to them. Children are not wicked as some of our forefathers thought, yet they are in a certain sense young savages. They have inherited most of what they are from a remote ancestry which knew nothing of line fences, refined manners, the alphabet, or the multiplication table It follows, therefore, that the immediate interests of children are those of the savage and not those of the twentieth-century adult If this were not true, there would be no need for a teaching profession. The children would learn all that they need to know without teachers

As matters stand, however, teachers have very important things to do The most important task of all for the teacher is to begin with the interests of the children as they are and gradually to widen and direct them until they include everything which humanity in a slow and painful evolution has found to be worth while. This is in itself a tremendous task, but it can be done. Children's interests develop and spread with marvelous rapidity when the right environment is provided To provide this en-

vironment is the function of the teacher, and it is here that she can attain unto a true and genuine expertness

The technique for the application of the law of use has been almost common knowledge for generations But we have no established technique for the application of the law of effect. Much, however, has been written on this topic in the last quarter of a century and we are learning many things. The process of applying the law of effect is often called "motivation" A further consideration of the details of this process must be postponed to the next chapter. For the present we shall proceed to a discussion of a few of the general principles involved from the standpoint of corrective arithmetic.

Some general principles In the first place, the teacher must remember that arithmetic was of comparatively little use to the savage From this it is suggested that even the brightest pupils inherit very little in the way of immediate interest in numbers. That is why the teacher of the first grade must busy herself in an effort to awaken such an interest through the manipulation of actual objects and through participation in group activities which involve quantitative relations. This first work is of fundamental importance, however, because without it the *law of effect* will work *against* the *law of use* instead of *with* it. Every effort must be made to make the child interested in learning the number facts which he must learn in order to take his proper place in organized society The facts referred to here are given in detail in the Appendix. The problem in hand is to interest the children in these facts or, in other words, to motivate their study.

Intelligent motivation grows out of an accurate knowledge of what satisfies children and what annoys them Children are annoyed by harsh words, mournful countenances, too much or too little heat, glaring sunshine,

uncomfortable seats, squalid surroundings, too much confinement in one position, foul atmosphere, physical pains, malnutrition, and the like They are satisfied by kind words, smiling faces, moderate temperatures, appropriate light, comfortable seats, pleasing environment, freedom of movement, pure air, and bodily comfort The first task of the teacher is to banish the annoyances and provide the proper satisfiers

The teacher must also remember that many satisfiers are not conducive to progress in arithmetic To watch a circus parade or a dog fight is intensely satisfying to children, but it is not conducive to the learning of arithmetic. Banish, therefore, so far as possible, all irrelevant satisfiers along with the annoyers Provide for the correct operation of the law of effect if you expect your teaching to be a success. A great amount of corrective work can be done by observing this one principle.

Fatigue

The law of use and the law of effect are the most general of all the laws of learning in their application. The other laws are, however, of great value in special situations Teachers often need to have accurate information concerning fatigue In this connection the most important thing to remember is that most of what is called fatigue is not fatigue in reality. It was once thought that human beings were like storage batteries and that rest and sleep were for the purpose of recharging them with energy. According to this doctrine, it was held that the early morning was the best time for work The hours after the evening meal were likewise regarded as the poorest because by that time nearly all of the energy accumulated during the previous night would be exhausted For the same reason it was recommended that difficult subjects like arithmetic should

come the first thing in the school day before much of the store of energy was exhausted.

It was asserted by others that fatigue was caused by the accumulation of chemical poisons in the blood. Sleep and rest according to this theory was for the purpose of cleansing the blood. In the morning the blood would naturally be more pure than in the afternoon. So we arrived by another route at the conclusion that the most difficult subjects should be taught first in the day.

Boredom

At present both of these theories are in the discard. They were intended to apply for the most part to physical fatigue, and it was assumed that mental fatigue could be accounted for in the same manner. This assumption has been seriously questioned It is a matter of common knowledge that many excellent students do their best work just before they retire instead of just after they get up Of more importance still is the distinction which has been made between fatigue and boredom Boredom affects both men and animals, and its effects are so similar to those of fatigue as to be almost indistinguishable from them. There was an old saying among the farm hands that no one could ever work a mule so hard that he would not run and kick up his heels when it was necessary to stop the work in the fields and hasten to the farmhouse to escape the oncoming storm. Children are much the same. Who has not seen a room full of apparently almost exhausted children suddenly transformed by the sudden ringing of the fire-gong? The truth of the matter is that healthy children in school seldom if ever feel the sensation of mental fatigue On the other hand, some children are often unutterably bored in almost every school.

The curve of average achievement has some fluctua-

tions in it during the day, but these fluctuations are relatively insignificant and are due for the most part to boredom All of this means that the important thing for the teacher to do is to prevent boredom. Don't keep the children at one thing too long Practice periods upon such material as that given in the Appendix of this book should never exceed two minutes unless the practice occurs in connection with a game. Even then the extremely long practice period is to be avoided Fifteen minutes at a time will be an abundance in the primary grades even for the game. As a general rule, always change the work as soon as the children appear to be tired In ninety-nine cases out of a hundred they are bored and not fatigued.

Multiple Response

Other principles of learning are of interest because they explain many of the errors listed in Chapters IV and V. Consider the juggling with figures which is so common when children attempt to solve reasoning problems which they do not understand When a child or an animal finds itself in a difficult situation from which it must escape, it employs, one after another, all of the reactions which it is capable of making Recently a certain man who was just learning to drive his car, which was of a well-known and widely used make, had the disagreeable sensation which comes from finding one's self stalled on a country road. He knew very little about automobiles, but he got out and began to manipulate various parts of the car in a vague and indiscriminate manner, but without success Finally in despair he gave the car a good shake, climbed in, and started off without further difficulty He does not know why he succeeded, but it is safe to assume that he will do some more shaking next time his car stalls

The foregoing is an excellent illustration of the manner

in which many pupils attack their reasoning problems. Any of us in a totally unfamiliar situation would react in a similar manner. The trouble is always due to a lack of prerequisite experience. Wild answers and juggling with figures are always reliable signs. It is a natural human reaction and always calls for a specific type of treatment. Reasoning is conditioned upon judgments and judgments are in turn conditioned upon experience. To attempt to teach reasoning without previous experience and without previous practice in forming judgments is sure to be futile.

The Law of Analogy

Another cause for many errors in arithmetic is the operation of the law of analogy. When men or animals meet with a situation which is partly new to them, they are likely to react in the same manner as they do to the old situation which is most like the new one. Thus children who have had practice in the addition of numbers in a single column when brought suddenly face to face with a situation like $\begin{array}{r}26\\ \underline{39}\end{array}$ will often get 515 for an answer. In like manner, if they are asked to multiply 3 ft. 6 in. by 3, they will get 10 feet and 8 inches because $36 \times 3 = 108$. Such results are invariably due to the fact that the child is not conscious of the new elements involved. The remedy for the situation is emphasis upon this new element.

Analysis

The process by which children are made conscious of the new elements in their work is called analysis. Practice in analysis has three phases: (1) identification of the new element; (2) practice in distinguishing the new ele-

ment from the old; (3) identification of the new element in various situations [1]

Identification of the new element In the first of the two examples given above, the new element is the fact that the numbers in one of the columns add up to more than 9 In the second example, the new element is the presence of the words "feet" and "inches." Point out these new things very carefully to the children.

Practice in distinguishing the new element from the old In the first example the child knows how to react to exercises of the type $\begin{array}{r}54\\ \underline{43}\end{array}$ They are familiar. But exercises of the type $\begin{array}{r}26\\ \underline{39}\end{array}$ are not so familiar. In the second example the child is familiar with exercises of the type "multiply 36 by 3," but is not familiar with the type "multiply 3 ft 6 in. by 3." In both cases practice is needed to enable the pupil to know whether the particular exercise which confronts him belongs to the old or the new type. If it is of the old type, he knows what to do. If of the new, he has only one additional thing to remember.

Identification of the new element in various situations. Children need practice in carrying and knowing when to carry under various situations The addition of digits other than 9 and 6 (as in the above example) makes carrying necessary. Carrying must be done in situations like the following:

$$\begin{array}{r}14\\23\\\underline{64}\end{array}\qquad\begin{array}{r}192\\26\\\underline{31}\end{array}\qquad\begin{array}{r}98\\76\\\underline{32}\end{array}\qquad\begin{array}{r}72\\4\\\underline{183}\end{array}$$

Practice is needed in the carrying of 2 and 3 as well as 1,

[1] For further information on this point see Thorndike, E L, *Psychology of Learning*, pages 37 ff. New York, Teachers College, Columbia University, 1913.

and in multiplication as well as addition In like manner the child must learn to react to the new denominate number element in connection with yards and feet, quarts and pints, pounds and ounces as well as with feet and inches.

Inference

Another principle of learning the neglect of which causes many errors is "the principle of inference." The ability to infer is one of the few things which definitely distinguish human beings from lower animals. The ability to size up what happens under certain situations and to infer what will happen under others is of tremendous practical importance. It is the only basis upon which we can plan intelligently for the future and it is fundamental to inductive reasoning. Trouble arises because little or nothing is known concerning the ability to infer which children possess at different ages One of the simplest forms of inference is the expectation that whatever is customary and satisfactory will remain so We all believe that next summer will be hot and next winter will be cold because summers have always been hot and winters cold No one would care to step off of the roof of a high building because it has always been customary and satisfactory to avoid such actions. Now, it is entirely normal for children who have been getting correct answers by adding to continue to add even when the type of exercise is radically changed. The same is true to a less degree of subtraction, multiplication, and division This accounts for the very large number of errors which children make through the use of the wrong process The way to correct this type of error is to emphasize more the analysis of new elements and the presence of changed conditions After this is done the teacher should not fail to provide abundant practice in mixed

exercises which involve all of the fundamental operations in simple form

The worst mistake, however, has been to expect from little children greater powers than they really possess Few children are able to infer that $4 \times 3 = 3 \times 4$, that 16 and 9 are 25 because 6 and 9 are 15, that since you multiply the length by the width to find the area, so you divide the area by the length to find the width Such ability is simply beyond the children at the ages when the above facts are expected to be learned. Under such circumstances there are only two sensible types of procedure We must find a new method of presentation or postpone the consideration of the facts until the pupils are old enough to infer correctly concerning them. As a general rule, proceed very carefully in those exercises which require inference on the part of the pupils They are in all probability much less able to infer than the teacher thinks The principle of inference has already been discussed at some length under the name of transfer of training The reader should read pages 11 ff again at this point After all, the best corrective measure under actual schoolroom conditions is to reduce to a minimum the number of situations in which children in the primary grades are expected to use inference in the solution of their problems and exercises.

Forgetting

One of the most discouraging things about teaching is the fact that so much of the work has to be done over again. Children forget what they learn. At the end of the school year they seem to be well up in their work; but after the summer vacation the teacher often receives looks of blankness and dismay when she asks questions which seemed easy to the children at the close of the pre-

vious session. We are all forgetful creatures Who would want to remember everything, anyway? The advantages of forgetting probably far outweigh its disadvantages because, though we forget useful things along with the rest, we can easily relearn that which is worth remembering. Here is the basis for the insistence upon the use of inventory tests at the beginning of the new school year Let us find out first of all how many valuable things the children have forgotten so that we can attend to these first.

Relearning

The forgetfulness of children, however, is not a hopeless condition Valuable corrective measures are possible. They are based on certain established facts In the first place, permanence of learning is possible only at the price of frequent relearning One out of every five men who entered the army in the recent war could not read a newspaper intelligently This was not because they had never learned to read They were simply "out of practice" In like manner, it is probable that very few adults who, twenty-five years ago, were star pupils in the Greek class, would care now to attempt to read at sight a passage from Herodotus or to conjugate a Greek verb They, too, are "out of practice." If such is the case with highly intelligent adults, the teacher should not be greatly disheartened when the little ones get "out of practice"

In the second place, that which has been once learned well and then forgotten can be relearned in an extremely short time. A very small amount of drill in silent reading would have changed the situation among the army recruits very decidedly for the better. Likewise a few minutes' review of the Greek grammar and the Herodotus book will work wonders for the star Greek pupil Chil-

dren will profit tremendously from a small amount of intelligent review upon that which they knew last year.

How to Improve the Memory

In the third place, a poor forgetter is a good memorizer. This brings up the question of how to improve one's memory and how to improve the memory of one's pupils. Educational psychology has help for us on this point. Some of the facts are as follows:

(1) The human brain is in many respects like a very complex telephone system The long and slender nerve cells correspond to the wires. Some of the cells lead inward from the eyes, ears, and other sense organs to the brain, while others lead outward to the hands, feet, vocal organs, and the like Still other cells act as connections within the brain between the cells which bring in the sensations from the sense organs and those which carry out "messages" to the muscles The teacher is to some extent at least like the switch-board operator whose business is to make and keep the right connections.

(2) The process of remembering includes four phases — the formation, retention, recall, and recognition of these connections

(3) Connections which are made for the first time are likely to be more permanent.

(4) The same is true of connections which are accompanied by vivid or striking feelings

(5) Single connections are more permanent than those of multiple type

(6) Connections are remembered best just as they are formed

(7) Recall is strengthened when the connections to be remembered are associated closely with situations like those which the child will meet most frequently during the vacation and after life.

The foregoing facts are so important that they merit further discussion Every teacher should be familiar with the four phases of remembering, namely, the formation of connections, their retention, recall, and recognition. She should be on the lookout for any sort of device which will tend to strengthen connections She should remember, too, that children grow at different rates and that there are tremendous variations in their ability to form and retain the connections which the teacher is striving to make. If we may return to the illustration of the telephone switchboard, we find some children whose minds are similar to an imaginary board in which the plugs are continually slipping out of the holes into which they are placed Other children have minds like a switchboard in which the plugs can be pushed in only with great difficulty. When they are once in, however, there is little danger of their falling out In other words, some children forget rather quickly almost everything that they have learned Others learn very slowly but retain well. These are the two extremes, however The average child learns with comparative ease and remembers reasonably well, provided the material is not beyond his grasp The ability to retain seems to be largely inherited. No corrective procedure is known which can improve retention when it is lacking The only hope is to wait for the child to grow.

While little can be done to increase the child's power of retention, much improvement can be made in the methods which promote memorizing Present the new element as vividly as possible Things are vivid to children when they appeal to fundamental interests and instincts. It is not possible within the limits of this book to discuss in detail what these fundamental interests and instincts are, but it is worth while to give at least one illustration. Chil-

dren are strongly individualistic. They are greatly interested in their own well-being. Anything that enables them to excel or to promote their own interests is very satisfying to them. On the other hand, anything that interferes with their personal success or detracts from their reputation among their fellows is thoroughly annoying All children like to engage in activities in which they can win and dislike anything that brings upon them, even in the slightest manner, the scorn or ridicule of their fellows. These instincts are crude, primeval forces which, like the lightning, do damage when undirected, but are very valuable when controlled properly Wise teachers know how to manipulate these forces In the teaching of primary number, the desire of the children to win is often used As soon as pupils find out that the number combinations which are being presented for the first time to-day will help them to win the game to-morrow, they become more interested, give more attention, and receive more vivid impressions If, when the next day comes, some child has forgotten and fails to win, he experiences the discomfort which comes to us all when some one passes us in the race. He will tend to remember the cause of his failure and see to it that the accident does not occur again. All of this promotes vividness, and vividness should accompany the forming of the connections on which learning is based

The author recently visited a good teacher on the day on which she was introducing carrying in addition for the first time. When the first sum turned out to be 10 or more, she called attention to the fact that the sum was really one 10 and so many units She reminded the children that they were to write the number of units only beneath the units column and instructed them to write the one 10 at the top of the 10's column Then she said, "There are some little children who forget to write what

is to be carried at the top of the next column, but I am almost sure that no one in this room will ever be so foolish as that." The result was that every child made a special effort to avoid this particular mistake because no one wanted to appear foolish. This is another illustration of how vividness of early impressions may be secured

Multiple connections are difficult to form and difficult to retain. For this reason it is a bad policy to teach number connections and rationalizations concerning them at the same time. Such devices as the use of bundles of sticks to explain carrying or borrowing; introducing the fours and eights as combinations of twos, trying to explain why you invert the divisor; attempting to teach the Austrian method of subtraction along with some other method; introducing such forms together as 4 is what part of 6? what part of 6 is 4? what per cent of 6 is 4? etc. — all of these are open to the objections stated above. To attempt to present more than one thing at a time is like trying to do two things at once. The probabilities are that neither will be done well.

Fundamental Number Facts Must Become Automatic

There is yet another argument to show the futility of trying to teach more than one thing at a time When a child fails to respond to a number connection, many teachers urge the pupil to *think about it* They attempt to secure the correct response by reasoning. This is objectionable on several grounds First, the other pupils are being deprived of time which should have been spent on them. Often they are bored almost to tears waiting for some retarded child to do successfully that which is perfectly easy to them but which the slow child is unable to do at all. Second, memorizing combinations and reasoning

about them are two quite different things and should not be taught together. Third, the procedure is of no value in real life. No one wants to have to stop and reason about 9 and 6 when he needs that combination.

Suppose a girl goes into the city and sees a very attractive hat in a store window. Suppose also that the girl needs the hat quite badly As she stands there her thoughts are probably something like this: "I need the hat. It will be very becoming to me All of my friends will remark about how well I look Let me see, I have so many pounds of butter and butter sells at so much per pound I can sell so many eggs at so much per dozen"; and so on. Then she begins multiplying and adding. Surely that girl should not be expected to reason about any particular number combinations which she is using. All of her thoughts are and should be centered on the hat.

The truth of the matter is that we must teach the fundamental number facts so well that the pupils will respond to them correctly *even when they are thinking or reasoning about something else.* The use of numbers and reasoning concerning them almost never occur together in life Our aim should be so to subject the number combinations to routine that reasoning will be set free to occupy itself elsewhere Reasoning about primary numbers is therefore just the thing which we should *not* do See to it, therefore, that the number facts are presented as simple facts to be learned and remembered After they are well mastered there will be plenty of time to practice their use in connection with reasoning

Space is available here merely to repeat what has been said several times before concerning the transfer of training It is not safe to expect most of the children to remember connections in other settings than those in which they were first formed. Many a child who has learned 9

and 7 will be confused when he first meets with 29 and 7. Do not expect much from this sort of transfer and you will at least avoid being disappointed

The recall of useful connections is strengthened by associating these connections with situations which occur frequently in life. For this reason it is an excellent idea to teach number combinations in connection with games which the children are likely to use during vacation and playtime. This is the chief justification for the use of such games as hopscotch and dominoes during the school hours In like manner, wise teachers will attempt to tie up the finding of per cents with the calculation of baseball standings.

Habit Formation

The mastery of primary arithmetic and the use of corrective arithmetic are both closely related to the formation of habits. In the minds of many people the word "habit" means something bad, but there are multitudes of good habits also. To see 7×9 and always to say, or be ready to say, 63 is a good habit. The steps involved in the formation of habits have been covered rather completely heretofore, but a summary will not be out of place here. The steps to be taken are as follows:

1. Get the child to *want* to form the habit. In other words, always observe the law of effect. Motivate your work.
2. See that each child knows just what he is to do This has been discussed under the subject of analysis A copy of the inventory test which the child has taken will usually furnish convincing evidence of what remains to be learned.
3. Practice the new elements frequently and persistently but with short practice periods

4. Avoid wrong responses as much as possible The most important need which the teacher of arithmetic has is a means of diagnosis which will enable her to avoid allowing the children to practice errors or allowing them to practice connections which are already well formed
5. Reward the right sort of practice in a suitable manner. There are various ways of doing this. A very good one is to keep a record of the scores made on successive practices, so as to show the amount of improvement Encourage the children to try to beat their own past records Nothing is more stimulating or satisfying than to be able to convince one's self that one is getting somewhere. Many other devices of this sort are used by our best teachers. Some of them are listed in the next chapter.
6. Use the inventory tests frequently to discover what the children are forgetting. Provide additional practice where it is needed.

Questions

Special mention should be made at this point of a few questions which have recently received much attention from those who are interested in the teaching of arithmetic.

Question 1 Should the pupils be required to check their work in every case or should the answers be provided in the textbook? The law of effect requires that the child know just as soon as possible whether or not he has succeeded This is a splendid argument in favor of supplying the answers at once to the child On the other hand, we are told first, that children will juggle figures in order to obtain the answer when they know what it is; second, that children will work only to obtain the answer and not from sheer love for the task; third, that we are

preparing children to solve problems in real life for which the answers are not known and will therefore not be available. The argument concerning the juggling of figures will appeal to the lazy teacher Pupils juggle figures to an alarming extent, even when they do not know the correct answer The teacher cannot escape her responsibility on this point It is her business to see that the pupils *do not* juggle figures

The statement that children will work merely to get the answer and not for the love of the task is scarcely worthy of notice What sane person would ever want to solve a problem or work an exercise in arithmetic for any purpose other than to obtain the correct answer?

The third argument is worthy of more serious consideration Children must be taught how to check their answers. This is particularly valuable as a corrective device in common fractions and in the division of decimals Checking must not be emphasized, however, to the extent of boring the pupil This is particularly true in column addition Most children do not object to adding upward but most of them detest adding downward after they have added upward. The use of checking is particularly harmful when used too soon Children who have not mastered column addition, for example, will be sure to make mistakes either in adding upward, in adding downward, or both. The result will be that the child will seldom if ever get his exercise to check To practice something continually and fail continually is extremely discouraging Practice of this sort makes the pupils imperfect rather than perfect

To avoid the foregoing intolerable situation, it will be necessary to furnish the correct answers until the exercises are well mastered. Then introduce and teach checking as a separate subject in the same manner as other new elements are introduced and taught.

Question 2 In beginning column addition, shall the children be taught to add upward or downward?

The most common practice is to teach the upward addition first No great amount of scientific data is available on the subject but that which we have indicates that the question is of small importance. Neither method can claim any significant advantage over the other. Under such circumstances it is preferable to follow the established usage Teach adding upward first, but do not forget to teach adding downward as a means of checking

Question 3. How shall subtraction be taught?

Two methods are available One is known generally while the other — the Austrian method — is not so well known The latter method has received ardent support, however. Its advocates claim that it greatly decreases the work of the teacher because, by it, subtraction becomes merely an application of addition instead of seeming entirely new to the child It is also asserted that the Austrian method is the one in common use in the making of change.

The latter argument is the result of loose thinking and is not worth considering. Suppose, for example, you have purchased something for 55 cents and you hand the merchant a dollar. He hands you your change, saying, " 55 and 10 and 65 and 10 and 75 and 25 is one dollar." But this is not the procedure of the Austrian method It would be as follows:

$$\begin{array}{r} \$1.00 \\ \underline{55} \end{array}$$

"Five and five are ten. Put five in units' place Remember to carry your one and add it to the five in the tens' place, 6 and 4 make 10. Write the 4 in the answer in tens' place" It is evident, therefore, that none but the most superficial observers will use this argument.

The first argument seems sound, but some recent results

indicate that children who have been taught by the Austrian method cannot subtract any better than those taught in the ordinary way[1] Until further evidence is forthcoming, we must conclude that we are not justified in abandoning the old method

Question 4. How shall borrowing in subtraction be taught?

Two methods have been widely used In one case you subtract what is borrowed from the next figure in the minuend; in the other you add one to the next figure in the subtrahend The main argument for the first method is that it prevents children from having to think of addition when they subtract This is said to be a simplification of the task and therefore desirable. The method of subtraction throughout causes trouble, however, in cases like

$$\begin{array}{r} 10000 \\ \underline{7879} \end{array}$$

Here the child has to go through a series of four subtractions before he can say 9 from 10 leaves 1. He must then remember that instead of having 1000 left in his minuend, he has 999 This is extremely difficult for many children, as is shown in the frequency of error in this sort of exercise.

The practice of adding what is borrowed to the subtrahend avoids trouble with zeros in the minuend, but it tends to complicate matters by mixing addition with subtraction.

Both methods are open to objection from the moralist In the first you borrow and fail to pay back, while in the second you pay back what has really not been borrowed The argument of the moralists is utterly worthless, however, because it is based upon a totally erroneous conception of childhood. Children are not little adults At the

[1] Mead, Cyrus D, and Sears, Isabel, "Additive Subtraction and Multiplicative Division Tested." *Journal of Educational Psychology*, 7 261–70

age when they are taught to "borrow," they are quite unable to sense the subtle logic of the situation Little children who feel glorified when they ride a stick horse or who revel in dolls with cracked heads or no heads at all are not going to be corrupted or morally injured in the least through the use of "borrowing" in subtraction. For this reason no apology is offered for use of the terms "borrow" and "pay back" in this book They are not logical or moral in the strictest sense, but they are well known by all and quite satisfactory from a practical point of view

The conclusion is that, as in the case of adding upward and downward, neither side can claim any great advantage. The same is true also concerning the question of whether in carrying one should write what is to be carried at the top or at the bottom of the next column One thing, however, is true and extremely *important*. *Be consistent* in what you do If you start one of these methods, carry it through. Use the same method used by your predecessor on all doubtful points as far as the same children are concerned Children who have partly learned one method of procedure should rarely if ever be taught another method by the next teacher. It is better to use a poor method well than two good methods poorly.

Question 5. When should the tables of addition, subtraction, multiplication, and division be taught *as such?*

The answer is, "Not at all." The number facts can never be used in the order in which they occur in the tables. Tables are merely summaries of what has been learned. They are of value as a means of *remembering* what has been learned They are useless as methods of *learning* If used at all, they should come in at the end of the third grade as a means of summarizing what has been learned.

This chapter would not be complete without mentioning

again the principles of economy in teaching discussed in Chapter II They are as follows·

1. Apportion the amount of drill to the difficulty of the task.
2. Teach first and most completely that which is most used in life.
3. Teach what the pupils do not know. Do not teach the things they already know
4. Do not try to teach things which the child could not learn even with the best teacher who ever lived
5. Always do your best to make pupils want to learn what they should learn.

CHAPTER IX

GAMES AND DEVICES

THE teacher who wishes to improve her work through corrective arithmetic will use an inventory test to find out just what she needs to teach. She will then strive to teach what is needed through the use of the best possible methods It has been shown that connections made in company with vivid and pleasant feelings are more permanent and more likely to be practiced spontaneously Vivid and pleasant feelings are always associated with the exercise of some fundamental instinct The desire to excel and the tendency to play are fundamental and widespread instincts in almost every child It is very desirable, therefore, to tie up the child's number work with these instinctive tendencies Two dangers, however, must be avoided. The play or contest element in the game must not be too highly organized for the little children or too infantile for the larger ones.

GAMES

Games and devices for use with number work are plentiful on the market. No teacher needs all of them Therefore every teacher must select the ones which she proposes to use Some are much better than others The purpose of this chapter is to list and describe those which have been found most useful by school supervisors in Wisconsin. Both rural and city supervisors were asked to list five games which they had found most useful in the teaching of primary number. Two hundred and thirteen games were mentioned including repetitions. The types of

games mentioned with their frequencies in percentage form are given below

	Per Cent
Miscellaneous	21
Races	17
Post-office game	12
I am thinking of two numbers whose sum is	10
Circle game	10
Games with cards	7
Playing store game	6
Beanbag game	6
Baseball	5
Wading brook game	2
Matching game	2
Train game	2

Racing games An inspection of the above data shows that games involving a race of some sort are used more frequently than any other type The race element is introduced in a variety of ways A very common form is to write the combinations in a long row if presented in the column form or in a column if presented in equation form Two children start at opposite ends of the column or row at a given signal and call answers as fast as possible The purpose of the game is to see which child can get to the center first. In using this device the same two children should reverse their positions and call the answers again because one half of the column or row may be more difficult than the other half If a different child wins each time, the result should be called a tie

Children get tired of the same game played all the time. It is necessary, therefore, that the teacher have in mind numerous variations A small variation in the setting of a good game will revive the interest of the children remarkably. For example, let each child call the answers to all the combinations in the row or column. Keep

time for each one by observing the second hand of your watch. The winner is the child who does the required work in the shortest time If either of the children makes a mistake, the teacher says, "That is too bad You didn't know how much —— (giving the name of the combinations missed) are You will have to study a little more on —— (naming the combinations missed) before you can win in this game" For these exercises absolute accuracy should be the standard for winning Maintain this standard if you can, but do not bore the children in order to do so

Children in the second grade enjoy a variation of this game in which the teacher runs along with her finger just *behind* the combinations whose answers the child is naming, saying, "Hurry, or I'll catch you," "Don't let me win," and the like.

Young children enjoy playing at picking the apples before the frost comes, gathering in the clothes before the rain, rescuing the guests from the burning hotel, and the like. Of course, the apples, articles of clothing, guests, etc, are the particular combinations which each child needs to practice The worst objection to the last game is the drawing of the apple tree, clothesline, etc., which must be done by the teacher before the game begins The drawing is worth while, however, particularly when the teacher has some facility at it.

The games so far listed have been of the individual type They are particularly useful among little children In the upper grades, say from the fourth up, group games become more and more useful For the group games, two leaders are appointed who "choose up," thus dividing the school or class into two groups A representative from each group is selected to take part in each race The different races or "events" vary in character. A common

type is the circle game Suppose, for example, that the teacher wishes to give practice in the more difficult higher decade additions given on page 164 of this book The number 19 may be written in the center of a circle while the numbers 0 to 9 inclusive are written in mixed order around the edge Two circles of this sort are provided They may be joined to form a bicycle. A child is chosen for each circle and both start calling or writing sums at a given signal The score may be either the number correct or the time required to give a set of entirely correct answers. The score for each event is added to see who wins the "meet." For the next pair of children the requirements may be changed by merely substituting another number for the number 19 The same device can be adapted to the teaching of any of the combinations where the first number is always the same

There are many other group games which permit the use of entirely different combinations each time. An example of these is the ladder game. Two ladders are drawn with combinations for rungs Two children "climb" the two ladders by giving the answer to each combination starting from the bottom. The scoring is the same as that for the circles

Another device — one which will interest little children — is to make two series of circles on the floor representing stones. Two pupils then attempt to "wade the brook" If they miss a combination, they have fallen into the water. Another form of discomfort when a combination is missed is the requirement that the child who missed the combination step back two steps. Any of these games may be used in relay form, thus securing drill for every child in the group when his turn comes The total score for the group is the sum of the scores made by the individuals composing it.

The use of the inventory tests as practice exercises has already been mentioned[1] Standards of speed and accuracy are available in the Courtis Manual[2] and in many of the newer textbooks The pupils can thus measure themselves in time and accuracy against each other as individuals, or as groups, or against the standard scores.

Another very helpful device is to keep the daily scores so that each child can measure himself *against his own past record*. This allows the slowest pupil a chance If he can't beat any one else, he can at least beat himself.

Post-office game Another type of game is used quite widely by the supervisors who reported to us. One of its forms is known as "the post-office game" A large number of cards with combinations on them are given to the child who is to be the postman or the postmaster. If it is the postman, he puts his cards in a satchel or some sort of container and passes down the aisles. Each child has a number for a name. Suppose, for example, the postman comes to a child whose name is *eight* All of the combinations which have an answer *eight* are his letters and should be left with him by the postman. Thus the same child might get the "letters" 8×1, $\begin{array}{r}3\\+5\\\hline\end{array}$ $\begin{array}{r}15\\-7\\\hline\end{array}$ $48 \div 6$, $\frac{1}{2}$ of 16, and so on If a "postmaster" is appointed, the children go to his office for their mail. It will add to the interest if the teacher calls herself the government inspector It is her work to check up the postman or postmaster. She may appoint deputy inspectors among the children with the understanding that when she stands at a certain place in the room, the children are to play as if she were not there at all. At such times all understand

[1] See page 58

[2] Courtis, S A, *Teachers' Manual for Practice Tests in Arithmetic* Yonkers, World Book Company, 1920

that a deputy is in charge The penalty for a mistake on the part of the postmaster, postman, or deputy is a temporary suspension from office because "The people had letters in the mail and you couldn't find them. We shall have to get some one who can really hold this job down"

The "I am thinking" game In another form of the game the teacher or some pupil says, "I am thinking of two numbers whose sum is so and so." Some other child is required to name two such numbers. Other statements used with the "I am thinking" game are: "I am thinking of two numbers which give seven when subtracted"; "I am thinking of two numbers which give fifty-four when multiplied"; "I am thinking of two numbers which give six when one is divided by the other"; and so on. Any correct response may be counted or the teacher may allow the class to keep on guessing until some child hits upon the actual pair of numbers in the mind of the questioner. The last device is helpful in locating children who make incorrect guesses.

There is some argument pro and con concerning the value of the "post-office" and "I am thinking." The objection is that much of the practice is given in an indirect manner. The children *start with the answers and work backwards* This, of course, is seldom true in life Thus the drill is being given in a form in which it will not be used.

On the other hand, there are great advantages. The most important thing of all is that the children like the games very much and actually do become proficient in their number work by means of them. It is claimed that since the correct answer or response is the aim, we should present it first in order to take advantage of the principle that first connections are more permanent It is also argued that this form of drill offers a very useful method

of classifying combinations. In subtraction, for example, there are ten combinations which yield the answer 9, ten which yield the answer 8, and so on, including the much-neglected ten which give the answer zero It is said to be advantageous to learning when the groups are thus constituted The weight of the argument undoubtedly favors the use of these games. Some loss results, but the gain much more than compensates for it.

Two types of games have been discussed thus far The racing games are for the purpose of developing speed, while the post-office and " I am thinking " games aim chiefly at accuracy. The chief characteristic of all speed drills is the element of time pressure. They should never be used until accuracy is well established Many of the games and devices used for the development of accuracy may be used as speed drills by the mere limitation of the time element. The group contest plan can, of course, be used in connection with the exercises in the textbook. A representative from each side begins working simultaneously on the same exercise. As he finishes, he erases his work, takes his seat, and another pupil from his side comes forward and starts to work on the next exercise. If one of the contestants makes a mistake, he has to find it himself. While he is looking for it, all the other pupils on his side are impatiently watching him and urging him on. Such a situation may prove very valuable The rest of his group get some useful practice in self-control; while he himself experiences a pronounced sense of discomfort, the memory of which will be a source of no small amount of motivation for his future work.

One extremely important caution must be observed with this device, however The child who delays his side must always be made to know just *what* combinations caused his trouble. If this caution is not observed, the

whole device breaks down and may even become positively pernicious

Card games. A third type of game or device is the use of cards upon which the several combinations are listed The cards may be used in connection with the development of both speed and accuracy. Their greatest value, however, is as a means of presenting and fixing new combinations for the first eight or ten times. Their use in connection with the post-office idea has already been discussed. The latter game cannot be played, however, until a considerable amount of practice has been given with other forms of card drill The cards are a convenient means of presenting the combinations at the very beginning The teacher selects three cards which show the first three combinations to be taught. Some of the children will know the answers already Have the answers given and see to it that each child gets some practice in naming them The combinations may be written on the blackboard and the class may take turns in coming to the board, pointing to each combination, and giving the correct answer If a child misses a combination, he should be made aware of that fact and a record made showing which combination was missed. Another form of drill with the three cards is the flashing of each card before each child in turn. A limited use may be made of responses in concert. A great difficulty with this method is the tendency of the class to depend upon one or two bright pupils for their responses The wide-awake teacher will soon detect the well-known echo effect which this sort of response produces. To offset this, some teachers request the quick pupils to "hide their eyes" or stay out of the game for a while This is rather hard on the best pupils and may operate as a punishment for correct responses and thus violate the law of effect A better procedure is to

have correct responses from groups of the class. Even at best the concert type of procedure is of doubtful value

After the class period on the first day, the children may begin to use the cards as seat work The children are requested to think of the answer to each combination and prove that their answer is correct by means of pasteboard squares or other objects.[1]

The mastery of the combinations is distinctly a process of habit formation This necessitates many short practices with intervening periods No teacher can expect her pupils to remember the three combinations presented on the first day unless further practice is given For this reason it is advisable to teach the same three combinations the next day along with three new ones introduced in the same manner as in the case of the original three It is a good plan to keep the original three combinations for about five days. Thus, if the work starts on Monday, the three combinations introduced on that day will be practiced every day of that week On the following Monday they will not be practiced. On Tuesday the three combinations which were first presented on the preceding Tuesday will be omitted, and so on The amount of drill for the first week will therefore be · Monday, three new combinations, Tuesday, the same three with three new ones, Wednesday, the six old combinations and three new ones, Thursday, the nine old combinations and three new ones; and Friday, the twelve old combinations and three new ones. From this point on, the practice material will consist of fifteen combinations Each day three old ones will be dropped and three new ones added Each combination that is dropped will have been practiced for five days in succession This process should be kept up until all the new combinations for the grade or half-grade

[1] See page 74

have been practiced for five days in succession In the meantime these combinations must be practiced in as many other games and with as many other devices as possible in order to prevent forgetting

After combinations or number facts have been presented for two or three days in class, there are many seat-work devices which may be used to provide further drill A few samples of useful seat work will be briefly described. According to one plan, each child is provided with cards bearing all the combinations which have been learned — also with cards bearing all the answers The child is required to match the answer with the corresponding combination Even before combinations are taught, cards may be used as seat work. For example, one set of cards may show in printed form the words, "one," "two," "three," etc , another set the corresponding Arabic numerals, another the Roman numerals, and still another the corresponding dots. These cards may be presented in chance order The child's task is to match the cards properly. Further seat work is suggested on page 141.

There are numerous other games that may be played with cards by the class as a whole. Second-grade children like to play "street car" The chairs are so arranged as to represent a street car The children sit in the chairs and each child has a card upon which a combination is written. The conductor comes along to collect the fare. The children pay their fares by naming the combination which they have The game "pussie wants a corner" can be adapted to number work by means of cards. A combination card is fastened in each corner of the room A child stands in each corner The child who is "it" runs about trying to displace some one who has a corner. This may be done by beating some one to the

corner when the children change places. It is understood, too, that no one is allowed to occupy a corner who is found to be unable to give the combination in that corner. The person who is " it " may always get a corner by naming the combination in the corner before the child who is already in the corner can name it. In " hunting " the children search about the schoolroom or in the yard for combination cards and are allowed to keep all they find. The games " grab-bag " and " fishing " are quite similar, yet different enough to prevent monotony.

Playing store A fourth type of game represented in the results of the questionnaire is that which is for the purpose of supplying the children with the experiences which are prerequisite to abstract number work and arithmetical reasoning " Playing store " was mentioned most frequently. The articles in the store should be represented by pictures and there should be plenty of toy money of all the common denominations Credit accounts may be kept with certain definite periods for paying up. Plenty of opportunities for making and receiving change should be provided The characteristic of this type of game is the fact that the children play at real life situations in which the number element is an essential.

Baseball. A fifth type of game is that in which the number work is *artificially* attached to some game in common use in the community This type was represented most frequently in the questionnaire by the game of baseball. There are several ways to play the game The following is a sample:

The room is divided into two teams. One team is at bat, the other is in the field. The catcher stands in one corner of the room with the first, second, and third basemen in the other corners The pitcher stands near the center of the room. The remainder of the group are

fielders The batter is, of course, a representative of the other side He stands just in front of the catcher facing the pitcher The teacher is the umpire. She decides disputes, calls strikes, and furnishes the pitcher with balls. The pitcher throws a ball by asking the batter to give the answer to a certain combination He may use his judgment (within the limits set by the umpire) concerning what combination he shall use upon the batter. The batter must answer three successive combinations correctly in order to score one for his side. If the batter misses, the umpire calls, "strike one," and the combination goes to the catcher, pitcher, first, second, and third basemen and then to any of the fielders who can give it If no one answers correctly, the batter continues, but if some one "catches him out" another batter takes his place and so on until there are three outs after which the two teams reverse their positions

Any teacher who uses games of this type will soon become convinced that the children are interested. The danger is that the play element may entirely obscure the educational value For this reason it would seeem preferable to limit somewhat the use of this sort of game

Games for use out of school A sixth type of game is worth mentioning, although it occurred rarely in the results of the questionnaire It is the type which is most likely to carry over into vacation time and thus provide at least some practice in number work while school is not in session Hopscotch and dominoes have already been mentioned as examples of this class Dominoes can also be played to advantage by pairs of children, even while school is in session. Usually, in order to score, the child must play so as to make the ends of the line of blocks add up to some multiple of five Multiples of other numbers may, however, be used. The doubles are to be placed

crosswise so as to form bases for side lines of blocks. With two ends to the main line and one end of each of several side lines a great deal of practice in addition and subtraction can be provided

Practice Tests

In addition to the games and devices suggested above, every teacher should have the use of one of the practice tests now on the market Their purpose is to provide suitable practice material under scientific conditions. They also provide useful standards in speed and accuracy. The Courtis[1] exercises were constructed to make these standards available. The Studebaker[2] exercises are very similar, but they are presented in a form which is preferred by many. The Osburn[3] exercises are for the purpose of supplementing certain deficiencies in the older material based upon the notion that direct teaching of *all* the useful combinations is essential.

Seat Work

One further type of practice deserves more attention than has yet been given to it All teachers, and particularly all rural teachers, are sadly in need of suitable seat work for their pupils In recent years it has been recognized that pupils can learn a great deal by themselves if the material is presented in the proper form This is glad tidings for teachers in one-room rural schools and for teachers whose rooms are crowded There is a pressing

[1] Courtis, S A *Standard Practice Tests in Arithmetic* World Book Co
[2] Studebaker, J W *Practice Exercises in Arithmetic* Scott, Foresman & Co.
[3] Osburn, W J *Corrective Exercises in Arithmetic* Houghton Mifflin Co

demand for an improved technique in the presentation of seat work This demand is very much strengthened by another factor of tremendous importance An improved technique in seat work is the most promising relief from the grade system with its intolerable lock step If a procedure can be provided by which the child can learn by himself — even in spite of a poor teacher — the way will begin to open so that the bright child can be required to work at least hard enough to "stretch the traces taut," and the slow pupils can proceed comfortably and without fear of derision or unfavorable comparison. There is no greater need in American instruction to-day than for some means of individualizing instruction. An improved technique in seat work represents the best hope of improvement in this respect. The problem has been attacked in Winnetka, Illinois, in Los Angeles, California, and in Wisconsin The results in Wisconsin are still in the formative stage, but one or two things have been proved quite definitely in all three localities

First, better seat work is needed in all subjects. Since time is limited and since the principle of correlation of subject matter is well established, there is no objection to providing copies of seat work relating to more than one subject. Experience has shown that arithmetic seat work can be given to advantage in connection with silent-reading seat work Second, the busy teacher needs to have in her possession a number of copies of seat-work material each carrying its own set of written directions. Thus equipped, all that she has to do is to pass out the cards. Third, the work must be provided in such form that it can be very easily scored or graded Busy teachers must be relieved from the drudgery of grading complicated seat work Fourth, the exercises for seat work must be appropriately graded in difficulty and the teacher must

know what exercises to use with each child, how long to use each one, and in what order to use them

Finally, each card must carry, so far as possible, its own motivation It must have its own particular purpose and the child must be made conscious of this purpose as a specific goal which he is to attain He must be made to understand that as soon as he has reached a suitable proficiency on one card he is to proceed to the next The discerning reader will at once see that this leads directly into the heart of both curriculum reorganization and educational psychology. What is the relative value of subject matter? What are the bonds or connections which should be formed? *When* should they be formed and in what order? These are big questions and no one can answer them fully until a great deal more research work has been done in physiological, psychological, educational, and social fields. The exercises which follow are in a very crude form Their gradation is extremely tentative as yet The need is so great and the demand so insistent, however, that even crude ungraded exercises are worth considering until further improvements are possible.

Prerequisites

(To be read by pupil)

How well can you read? Children who cannot read cannot do well in school Can you read? If so, show me that you can by doing what it says to do on this card. Read this to yourself

A dog can sing.	No.
A bird can fly.	Yes

Here are two very short stories The first one has NO after it because NO makes it true. The second has YES after it because YES makes it true Now write the numbers from 1 to 20 on your tablet like this:

1

2

3

Now read the first little story below, and write after 1 on your tablet the word that makes this first story true. Then write after 2 on your tablet the word which makes the second story below true Go on in this way until you have done all the stories.

1. A girl likes flowers
2. You have four eyes.
3. A bird has wings
4. Flowers look like cows.
5. Does water run down hill?
6. I found flowers on the cat.
7. Men were once little birds.
8. A cat is as long as a horse.
9. The boy who ran was once a baby.
10. The cow will soon be a horse.
11. We all come here to eat corn.
12. The wind makes the leaves fly.
13. The garden is back of the house.
14. A cat is about as big as a house
15. When school is out we can go home.
16. I live in a house under the water.
17. I come to school to eat and drink
18. I had something to eat this morning.
19. The boy can see that the snow is blue
20. A cat has eyes. That is why she can see.

Slips containing the correct answers should be passed out so that the child can compare his answers with the correct ones. No child who fails to get fifteen answers correct should be expected to do any reading in connection with his number work All of the words in this exercise are in the vocabulary of the first grade with the exception

of the words "doing," "mark," "card," "numbers," "twenty," and "tablet" If the teacher has time, it will be worth while to give a little help to some of the children on these words The sentences themselves were selected from the Wisconsin Supervisory Tests in Primary Reading.[1] Other forms of these tests contain further material of this sort. The vocabulary out of which the tests were constructed is also given.

Seat Work for Second Grade

First sample

How Well Can You Count?

If you can count, you can answer the questions on this card See if you can get them all right Write your answers on your tablet. Be sure to number each answer

1 How many questions are there on this card?
2 How many sides has this room?
3 How many doors are there?
4. How many windows?
5 How many window panes in each window?
6 How many window panes in all?
7 How many seats are in the room?
8. How many children are in the room?
9. How many boys?
10. How many girls?
11. How many fingers, toes, and thumbs have you?
12. How many eyes have you?
13 How many eyes are in this room?
14. How many fingers and thumbs are in the room?
15. How many of the girls have ribbons in their hair?
16. How many boys have shoes on?
17. How many boys are wearing coats?

[1] These tests were devised by H W Kircher and are published by the Eau Claire Book & Stationery Company, Eau Claire, Wisconsin

18. How many girls have white dresses on?
19. How many brothers and sisters have you?
20. How many books have you in your desk?

When the children get through, they are to exchange papers and grade each other The grade or score is the number correct The teacher is to be called in only for the purpose of settling disagreements.

Further material of this sort may be obtained easily by collecting pictures from books and magazines. Write out questions concerning how many boys, girls, birds, frogs, etc, are in each picture The sheet containing the directions and questions is pasted on one side of a folder while the picture is pasted on the other side Old books and loose leaves are often very valuable in this connection. Lists of correct answers should be provided and given to the child as soon as he is through answering the questions The child's answers should be at least eighty per cent correct.

Second sample.

How Well Can You Measure?

(Materials needed — A yardstick and a footrule)

Use your ruler and yardstick to find the answers to each of these questions. Write the answers on your tablet.

1. How far is it across your desk?
2. How far is it from the front to the back of your desk?
3. How high is your desk?
4. How high are the windows from the floor?
5. How long is the schoolroom?
6. How wide is it?
7. How high is the stove?
8. How high is the blackboard above the floor?
9. How high is your seat above the floor?

10 How far is it around the room?
11. How wide are the windows?
12. How wide is the door?
13 How long is your reading book?
14 How wide is it?
15 How tall are you?
16. How far is it from your wrist to the end of your fingers?
17 How long is the chalk box?
18 How wide is it?
19 How high is it?
20. How long is your shoe?

The answers are scored by comparison with the answers given by others to the same questions. The children are to do the scoring as far as possible. Similar exercises should be provided with pint and quart measures and hand scales.

Third sample.

HOW MUCH DO YOU KNOW?

Children should be able to answer at least 15 of these questions before they are promoted from the second grade. How many can you answer? Write the answers on your tablet and number them If you do not know an answer, skip the question and go on to the next one.

1 How many inches in a foot?
2 How many days in a week?
3. How many feet in a yard?
4 How many things in a dozen?
5 How many pints in a quart?
6. How many ounces in a pound?
7. How many hours in a day?
8. How many quarts in a gallon?
9 Is a dollar more than ten dimes?
10. Is a peck less than a gallon?

11. Is a quart more than half a gallon?
12. Are ten days more than a week?
13. Is a nickel more than a dime?
14. Is six less than a dozen?
15. Is fifty less than thirty?
16. Is twenty-five more than 100?
17. Is V more than X?
18. Is VIII less than 8?
19. Do twelve eggs make a dozen?
20. Do twelve chairs make a dozen?

Pass out the correct answers when the children finish and let them do the grading

For Grade Three or Grade Four

(To be read by pupil)

Some children try to solve their problems without knowing what they are going to do Are you one of these? Each of these problems has add, subtract, multiply, divide after it. One of the words tells the right thing to do Write on your tablet the word which tells the right thing to do in each problem. *Do not work the problem.*

1. Apples are selling at seven cents each. I wish to buy four apples How much money must I spend?

Add Subtract Multiply Divide

2 My father sent me to the bank with $2.50 and told me to ask the banker for that many nickels. How many nickels should the banker have given me?

Add Subtract Multiply Divide

3. My mother gives me a fifty-cent piece and sends me to the store to buy a pound of tea. The pound of tea costs 45 cents. How much change do I receive?

Add Subtract Multiply Divide

4. My mother paid $12 00 for a bed and four dollars for a chair. How much money did she spend?

Add Subtract Multiply Divide

5. We sold six dozen eggs at 35 cents a dozen. How much money did we receive?

Add Subtract Multiply Divide

6. The children in my room are going to give a party. The expenses will be $1.50 There are thirty children in the room. How much will each of us have to pay?

Add Subtract Multiply Divide

7. My bicycle cost $22 50, my ball 25 cents. How much did I spend for both?

Add Subtract Multiply Divide

8 There are 18 boys in my room and 12 girls. How many more boys are there than girls?

Add Subtract Multiply Divide

Lack of space prevents the giving of more than a few samples of seat-work devices here It is hoped, however, that enough has been presented to enable the teacher to make up similar devices of her own They are particularly needed as corrective exercises to meet the type errors of the class.

The whole process of corrective arithmetic may be summed up in these words Find what needs to be taught in your grade and what should have been taught in previous grades Give an inventory test based on the work of the previous grades Note the errors and the names of the children who make them Plan your class and seat work on the basis of what the children need Busy teachers will have to rely to a large extent on the seat work. For this type of work it is hardly possible to have too much material The material must be in such form that it may be used by the children with a minimum of

help from the teacher. Clear directions and a definite means of scoring are essential. Finally it is very essential for every teacher to understand that the devices for seat work are of no value on the pages of a book. They *must* be available in loose-leaf or card form. Until the seat exercises are published in this form it will be necessary to obtain the copies in some other manner. Many teachers are mimeographing or multigraphing them When the copies are made, they should be put into such form that the teacher may keep them for use in future years. Seat work is the capsheaf of the corrective process. Without it all of the efforts spent in testing and diagnosing will be like grain which withers away before it ripens.

CHAPTER X

SUMMARY

The salient points which have resulted from the studies described in this book are · (1) the new meaning which old principles of teaching take on in the light of recent educational research, (2) the futility of our naive faith in transfer of training; (3) a clearer notion of just what is to be taught — its difficulty and importance socially; (4) the nature and causes of errors, (5) the inadequacy of available practice material; (6) the technique of educational diagnosis, (7) the identification and order of bonds to be formed; and (8) the use of educational inventories

Renewed emphasis has been placed upon the old principles of teaching because they are fundamental and will continue to be guides in our future progress Whenever we learn how to teach just a little better, with just a little less effort, we remove just that much friction from our educational machinery and the art and pleasure of teaching is accordingly enhanced. The breakdown of our former unconscious faith in transfer results in a large increase in the number of combinations to be taught. This does not necessarily mean an increase in the teacher's labor, however We shall in a large measure rid ourselves of that notorious thorn in the flesh which has resulted from the presence in the class of a number of children whose condition is hopeless For them the transfer did not take place. Transfer is closely related to the power to infer, and this in turn is a function of native intelligence. The slower children cannot make correct inferences. Whatever is taught them must be taught directly. This is

work for the teacher, but it is infinitely better than striving to teach by a method which is hopeless.

The teacher's burden will be lightened also because she will have a clearer idea of the end to be reached. The mere knowledge of just which combinations are to be taught will save a large amount of labor for the teacher and of hopelessness for the child

Teaching will become more expert and therefore more satisfying when the teacher knows the nature and causes of errors, when she is master of the technique of educational diagnosis, and when she has a clear notion of just what supplementary practice material is needed. We shall no longer permit little children to suffer from the discouragement which comes when they are expected to know that which is not to be found in their textbooks. We shall no longer compel them to encounter difficulties for which they have not been prepared.

In our teaching we shall proceed from the simple to the complex with an insight and wisdom which has not been known heretofore. Finally, we shall put teaching on a sound basis by making a careful inventory of just what each child already knows and just what he is expected to learn.

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APPENDIX

THE ONE HUNDRED PRINCIPAL FACTS WHICH MUST BE TAUGHT IN SIMPLE ADDITION[1]

THE EASIER GROUP

(1)	(2)	(3)	(4)	(5)	(6)	(7)
0	6	0	2	7	1	5
2	2	3	1	0	7	3

(8)	(9)	(10)	(11)	(12)	(13)
4	3	0	1	6	2
1	5	5	0	1	2

(14)	(15)	(16)	(17)	(18)	(19)
7	0	3	3	1	5
2	8	0	1	2	4

(20)	(21)	(22)	(23)	(24)	(25)
3	2	4	8	2	1
6	4	3	1	5	4

(26)	(27)	(28)	(29)	(30)	(31)
0	4	5	7	3	0
4	4	0	1	4	6

(32)	(33)	(34)	(35)	(36)	(37)
2	0	4	6	4	9
3	7	2	3	5	0

(38)	(39)	(40)	(41)	(42)	(43)
2	6	1	1	0	5
7	0	3	5	0	1

[1] Copies of this list may be obtained in quantities from the Public School Publishing Co. Bloomington, Ill.

(44)	(45)	(46)	(47)	(48)	(49)
2	3	4	1	8	0
0	3	0	1	0	1

(50)	(51)	(52)	(53)	(54)	(55)
1	5	2	3	1	0
8	2	6	2	6	9

THE MORE DIFFICULT GROUP

(56)	(57)	58)	(59)	(60)
5	3	6	7	8
7	9	7	4	9

(61)	(62)	(63)	(64)	(65)
5	6	8	4	9
9	4	6	9	7

(66)	(67)	(68)	(69)	(70)
8	9	7	7	9
3	4	9	3	3

(71)	(72)	(73)	(74)	(75)
9	3	9	3	9
6	8	5	7	1

(76)	(77)	(78)	(79)	(80)
5	9	6	9	4
6	2	5	9	8

(81)	(82)	(83)	(84)	(85)
7	8	8	6	7
5	7	4	6	8

(86)	(87)	(88)	(89)	(90)
1	5	4	5	9
9	8	7	5	8

(91)	(92)	(93)	(94)	(95)
7	8	4	8	6
6	5	6	2	8

(96)	(97)	(98)	(99)	(100)
6	8	2	7	2
9	8	9	7	8

THE ONE HUNDRED FACTS WHICH MUST BE TAUGHT IN SIMPLE SUBTRACTION[1]

THE EASIER GROUP

(1)	(2)	(3)	(4)	(5)	(6)	(7)
8	7	4	9	9	8	3
0	1	2	7	1	8	1

(8)	(9)	(10)	(11)	(12)	(13)
6	4	0	6	9	4
0	1	0	1	9	0

(14)	(15)	(16)	(17)	(18)	(19)
6	1	2	9	8	5
5	1	0	0	6	4

(20)	(21)	(22)	(23)	(24)	(25)
8	7	1	7	5	7
7	2	0	3	2	6

(26)	(27)	(28)	(29)	(30)	(31)
5	3	8	9	6	9
0	3	1	3	2	5

(32)	(33)	(34)	(35)	(36)	(37)
3	4	8	5	9	7
2	4	4	3	4	0

[1] Copies of this list may be obtained in quantities from the Public School Publishing Co., Bloomington, Ill.

(38)	(39)	(40)	(41)	(42)	(43)
8	5	4	6	7	9
5	5	3	6	4	8
(44)	(45)	(46)	(47)	(48)	(49)
6	9	8	9	7	3
4	2	3	6	7	0
(50)	(51)	(52)	(53)	(54)	(55)
7	8	2	5	2	6
5	2	2	1	1	3

THE MORE DIFFICULT GROUP

(56)	(57)	(58)	(59)	(60)
12	16	17	12	14
8	7	9	4	8
(61)	(62)	(63)	(64)	(65)
13	10	11	11	15
9	8	4	8	7
(66)	(67)	(68)	(69)	(70)
10	14	11	13	10
1	5	5	4	3
(71)	(72)	(73)	(74)	(75)
18	13	10	10	15
9	5	2	9	9
(76)	(77)	(78)	(79)	(80)
11	11	17	15	12
3	9	8	6	5
(81)	(82)	(83)	(84)	(85)
16	10	11	12	15
9	7	2	6	8

(86)	(87)	(88)	(89)	(90)
13	12	12	13	12
7	3	7	6	9
(91)	(92)	(93)	(94)	(95)
11	10	14	10	16
6	4	7	5	8
(96)	(97)	(98)	(99)	(100)
10	13	14	11	14
6	8	9	7	6

If the children are taught to "pay back" by adding one to the next figure of the subtrahend, the following combinations must be added to this list.

19	18	17	16	15
10	10	10	10	10
14	13	12	11	10
10	10	10	10	10

THE ONE HUNDRED FACTS WHICH MUST BE TAUGHT IN SIMPLE MULTIPLICATION[1]

THE EASIER GROUP

(1)	(2)	(3)	(4)	(5)	(6)	(7)
6	0	3	0	4	2	4
1	2	6	4	6	1	1

(8)	(9)	(10)	(11)	(12)	(13)
7	3	6	1	8	1
0	3	0	6	3	9
(14)	(15)	(16)	(17)	(18)	(19)
8	5	0	8	4	9
0	3	3	1	3	2

[1] Copies of this list may be obtained in quantities from the Public School Publishing Co Bloomington, Ill

(20)	(21)	(22)	(23)	(24)	(25)
1	7	1	1	0	7
3	2	4	8	6	3
(26)	(27)	(28)	(29)	(30)	(31)
0	3	3	3	4	6
8	4	8	5	5	4
(32)	(33)	(34)	(35)	(36)	(37)
0	9	0	5	2	0
9	1	7	4	7	0
(38)	(39)	(40)	(41)	(42)	(43)
2	2	3	3	2	3
2	6	2	1	0	7
(44)	(45)	(46)	(47)	(48)	(49)
7	1	1	0	1	2
1	5	7	5	0	3
(50)	(51)	(52)	(53)	(54)	(55)
5	4	1	2	6	2
1	4	1	9	2	4
(56)	(57)	(58)	(59)	(60)	(61)
5	8	0	6	1	3
2	2	1	3	2	0
(62)	(63)	(64)	(65)	(66)	(67)
5	2	9	4	4	2
0	8	0	0	2	5

THE MORE DIFFICULT GROUP

(68)	(69)	(70)	(71)	(72)	(73)
5	9	5	9	6	9
8	5	7	6	7	8

(74)	(75)	(76)	(77)	(78)	(79)
8	5	8	7	4	9
7	9	4	7	8	9

(80)	(81)	(82)	(83)	(84)	(85)
4	7	8	3	9	7
7	9	6	9	7	5

(86)	(87)	(88)	(89)	(90)	(91)
8	7	8	6	6	8
9	6	5	9	6	8

(92)	(93)	(94)	(95)	(96)	(97)
5	9	7	4	5	9
6	4	8	9	5	3

(98)	(99)	(100)
7	6	6
4	8	5

THE NINETY FACTS WHICH MUST BE TAUGHT IN SIMPLE DIVISION

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1)6	2)0	6)18	4)0	6)24	1)2	1)4	3)9
(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
6)6	3)24	9)9	3)15	3)0	1)8	3)12	2)18
(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)
3)3	2)14	4)4	8)8	6)0	3)21	8)0	4)12
(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)
8)24	5)15	4)24	5)20	9)0	7)0	1)9	4)20

(33)	(34)	(35)	(36)	(37)	(38)	(39)	(40)
7)14	2)4	6)12	2)6	1)3	1)7	5)5	3)6
(41)	(42)	(43)	(44)	(45)	(46)	(47)	(48)
5)0	1)5	7)7	4)16	1)1	9)18	2)12	4)8
(49)	(50)	(51)	(52)	(53)	(54)	(55)	(56)
2)10	1)0	2)16	3)18	2)2	8)16	2)8	5)10

(57)
7)21

THE MORE DIFFICULT GROUP

(58)	(59)	(60)	(61)	(62)	(63)	(64)	(65)
8)40	5)45	6)54	7)35	8)72	7)42	9)45	7)56
(66)	(67)	(68)	(69)	(70)	(71)	(72)	(73)
4)32	7)49	8)32	7)28	9)81	6)48	9)27	7)63
(74)	(75)	(76)	(77)	(78)	(79)	(80)	(81)
5)35	9)63	6)42	5)40	9)72	6)36	8)64	6)30
(82)	(83)	(84)	(85)	(86)	(87)	(88)	(89)
4)36	8)56	9)36	5)25	9)54	4)28	3)27	8)48

(90)
5)30

THE TWO HUNDRED AND TWENTY-FIVE COMBINATIONS WHICH MUST BE TAUGHT AS A PREREQUISITE FOR COLUMN ADDITION[1]

THE EASIER GROUP

(1) 33 and 5
(2) 13 and 6
(3) 31 and 7
(4) 13 and 2
(5) 34 and 1
(6) 13 and 5
(7) 10 and 5
(8) 38 and 1
(9) 31 and 5
(10) 26 and 1
(11) 31 and 8
(12) 11 and 7
(13) 23 and 1
(14) 30 and 6
(15) 34 and 3
(16) 11 and 8
(17) 20 and 6
(18) 34 and 2
(19) 20 and 7
(20) 37 and 2
(21) 30 and 7
(22) 23 and 6
(23) 31 and 3
(24) 37 and 1
(25) 23 and 5
(26) 33 and 6
(27) 10 and 7
(28) 22 and 3
(29) 33 and 1
(30) 20 and 8
(31) 12 and 6
(32) 13 and 3
(33) 22 and 4
(34) 17 and 2
(35) 33 and 4
(36) 10 and 8
(37) 34 and 4
(38) 22 and 6
(39) 11 and 3
(40) 31 and 1
(41) 33 and 3
(42) 13 and 4
(43) 11 and 2
(44) 21 and 8
(45) 17 and 1
(46) 30 and 9
(47) 22 and 5
(48) 11 and 1
(49) 23 and 4
(50) 30 and 8
(51) 10 and 9
(52) 34 and 5
(53) 11 and 4
(54) 20 and 5
(55) 31 and 4
(56) 26 and 2
(57) 13 and 1
(58) 22 and 2
(59) 11 and 6
(60) 33 and 2
(61) 23 and 3
(62) 22 and 7
(63) 26 and 3
(64) 31 and 2
(65) 12 and 7
(66) 20 and 9
(67) 31 and 6
(68) 22 and 1
(69) 21 and 7
(70) 23 and 2
(71) 10 and 6
(72) 11 and 5
(73) 25 and 3
(74) 10 and 2
(75) 35 and 3
(76) 12 and 5
(77) 15 and 2
(78) 10 and 3
(79) 21 and 3
(80) 24 and 1
(81) 35 and 1
(82) 30 and 2
(83) 15 and 3
(84) 10 and 4
(85) 21 and 6
(86) 25 and 2
(87) 20 and 4
(88) 21 and 5
(89) 32 and 7
(90) 12 and 1
(91) 24 and 4
(92) 16 and 2
(93) 20 and 3

[1] Copies of this list may be obtained in quantities from the Public School Publishing Co., Bloomington, Ill.

(94) 30 and 1
(95) 14 and 4
(96) 12 and 3
(97) 20 and 1
(98) 21 and 4
(99) 10 and 1
(100) 21 and 1
(101) 12 and 2
(102) 30 and 4
(103) 36 and 1
(104) 32 and 6
(105) 21 and 2
(106) 12 and 4
(107) 27 and 2
(108) 18 and 1
(109) 25 and 4
(110) 14 and 3
(111) 36 and 3
(112) 15 and 1
(113) 24 and 5
(114) 32 and 1
(115) 30 and 3
(116) 24 and 3
(117) 14 and 5
(118) 28 and 1
(119) 32 and 3
(120) 35 and 4
(121) 32 and 2
(122) 15 and 4
(123) 36 and 2
(124) 30 and 5
(125) 24 and 2
(126) 35 and 2
(127) 16 and 1
(128) 27 and 1
(129) 20 and 2
(130) 32 and 4
(131) 25 and 1
(132) 14 and 2
(133) 32 and 5
(134) 16 and 3
(135) 14 and 1

THE MORE DIFFICULT GROUP

(136) 19 and 5
(137) 15 and 5
(138) 17 and 5
(139) 29 and 3
(140) 23 and 7
(141) 15 and 9
(142) 17 and 3
(143) 29 and 7
(144) 13 and 8
(145) 26 and 9
(146) 22 and 8
(147) 14 and 8
(148) 29 and 9
(149) 19 and 6
(150) 23 and 9
(151) 13 and 7
(152) 17 and 4
(153) 19 and 8
(154) 17 and 9
(155) 29 and 1
(156) 26 and 7
(157) 16 and 8
(158) 22 and 9
(159) 26 and 5
(160) 11 and 9
(161) 14 and 7
(162) 23 and 8
(163) 19 and 4
(164) 24 and 9
(165) 29 and 5
(166) 25 and 7
(167) 29 and 4
(168) 15 and 6
(169) 25 and 5
(170) 21 and 9
(171) 29 and 8
(172) 15 and 7
(173) 19 and 2
(174) 28 and 8
(175) 29 and 6
(176) 14 and 9
(177) 19 and 7
(178) 17 and 8
(179) 26 and 4
(180) 27 and 9
(181) 17 and 7
(182) 16 and 9
(183) 17 and 6
(184) 13 and 9
(185) 25 and 9
(186) 24 and 8
(187) 18 and 9
(188) 25 and 6
(189) 26 and 8
(190) 19 and 3
(191) 28 and 7
(192) 25 and 8
(193) 19 and 9
(194) 26 and 6
(195) 19 and 1
(196) 28 and 9
(197) 12 and 8
(198) 15 and 8
(199) 29 and 2
(200) 12 and 9
(201) 16 and 4
(202) 18 and 4
(203) 14 and 6
(204) 28 and 2
(205) 27 and 3
(206) 18 and 7
(207) 27 and 4

(208) 28 and 5
(209) 24 and 6
(210) 16 and 6
(211) 27 and 6
(212) 18 and 8
(213) 16 and 7
(214) 18 and 6
(215) 16 and 5
(216) 27 and 7
(217) 28 and 4
(218) 24 and 7
(219) 18 and 2
(220) 28 and 6
(221) 27 and 5
(222) 18 and 3
(223) 27 and 8
(224) 18 and 5
(225) 28 and 3

THE ONE HUNDRED AND SEVENTY-FIVE COMBINATIONS WHICH MUST BE TAUGHT AS A PREREQUISITE FOR CARRYING IN MULTIPLICATION[1]

THE EASIER GROUP

(1) 25 and 3
(2) 72 and 7
(3) 10 and 2
(4) 81 and 5
(5) 35 and 3
(6) 40 and 2
(7) 12 and 5
(8) 15 and 2
(9) 81 and 6
(10) 56 and 2
(11) 10 and 3
(12) 72 and 1
(13) 40 and 5
(14) 21 and 3
(15) 64 and 4
(16) 24 and 1
(17) 81 and 4
(18) 35 and 1
(19) 30 and 2
(20) 15 and 3
(21) 10 and 4
(22) 21 and 6
(23) 25 and 2
(24) 48 and 1
(25) 20 and 4
(26) 63 and 5
(27) 45 and 1
(28) 21 and 5
(29) 40 and 1
(30) 56 and 1
(31) 32 and 7
(32) 12 and 1
(33) 81 and 7
(34) 54 and 2
(35) 24 and 4
(36) 63 and 1
(37) 42 and 1
(38) 16 and 2
(39) 81 and 2
(40) 20 and 3
(41) 64 and 2
(42) 30 and 1
(43) 40 and 6
(44) 14 and 4
(45) 42 and 2
(46) 12 and 3
(47) 20 and 1
(48) 14 and 1
(49) 21 and 4
(50) 40 and 7
(51) 10 and 1
(52) 56 and 3
(53) 72 and 4
(54) 21 and 1
(55) 12 and 2
(56) 42 and 5
(57) 30 and 4
(58) 54 and 3
(59) 54 and 1
(60) 36 and 1
(61) 32 and 6
(62) 21 and 2
(63) 12 and 4
(64) 81 and 1
(65) 27 and 2
(66) 72 and 5
(67) 63 and 3
(68) 18 and 1
(69) 25 and 4
(70) 63 and 6
(71) 42 and 4
(72) 14 and 3

[1] Copies of this list may be obtained in quantities from the Public School Publishing Co, Bloomington, Ill.

(73) 36 and 3
(74) 15 and 1
(75) 40 and 3
(76) 24 and 5
(77) 72 and 6
(78) 32 and 1
(79) 45 and 2
(80) 30 and 3
(81) 81 and 3
(82) 24 and 3
(83) 14 and 5
(84) 28 and 1
(85) 54 and 4
(86) 81 and 8
(87) 40 and 4
(88) 32 and 3
(89) 64 and 5
(90) 35 and 4
(91) 54 and 5
(92) 32 and 2
(93) 15 and 4
(94) 36 and 2
(95) 63 and 4
(96) 72 and 3
(97) 30 and 5
(98) 24 and 2
(99) 35 and 2
(100) 42 and 3
(101) 16 and 1
(102) 64 and 3
(103) 42 and 6
(104) 27 and 1
(105) 72 and 2
(106) 64 and 1
(107) 20 and 2
(108) 32 and 4
(109) 45 and 3
(110) 25 and 1
(111) 14 and 2
(112) 32 and 5
(113) 45 and 4
(114) 63 and 2
(115) 16 and 3

THE MORE DIFFICULT GROUP

(116) 16 and 4
(117) 35 and 6
(118) 18 and 4
(119) 36 and 4
(120) 54 and 8
(121) 14 and 6
(122) 28 and 2
(123) 45 and 8
(124) 27 and 3
(125) 56 and 4
(126) 36 and 7
(127) 18 and 7
(128) 49 and 1
(129) 27 and 4
(130) 63 and 8
(131) 28 and 5
(132) 49 and 6
(133) 56 and 7
(134) 24 and 6
(135) 48 and 5
(136) 16 and 6
(137) 49 and 5
(138) 64 and 7
(139) 27 and 6
(140) 18 and 8
(141) 16 and 7
(142) 72 and 8
(143) 16 and 5
(144) 36 and 6
(145) 18 and 6
(146) 48 and 3
(147) 27 and 7
(148) 28 and 4
(149) 56 and 6
(150) 36 and 5
(151) 45 and 7
(152) 24 and 7
(153) 36 and 8
(154) 18 and 2
(155) 45 and 5
(156) 28 and 6
(157) 64 and 6
(158) 35 and 5
(159) 48 and 7
(160) 27 and 5
(161) 18 and 3
(162) 45 and 6
(163) 63 and 7
(164) 27 and 8
(165) 49 and 3
(166) 48 and 4
(167) 54 and 7
(168) 18 and 5
(169) 48 and 2
(170) 56 and 5
(171) 49 and 4
(172) 28 and 3
(173) 48 and 6
(174) 54 and 6
(175) 49 and 2

THE TWO HUNDRED AND NINETEEN SUBTRACTION COMBINATIONS WHICH ARE PREREQUISITE TO SHORT DIVISION

THE EASIER GROUP

(1) 21 from 23
(2) 54 from 56
(3) 24 from 27
(4) 7 from 8[1]
(5) 40 from 47
(6) 18 from 19
(7) 20 from 21
(8) 12 from 17
(9) 54 from 55
(10) 12 from 15
(11) 7 from 9[1]
(12) 45 from 49
(13) 30 from 32
(14) 5 from 8[1]
(15) 45 from 47
(16) 30 from 31
(17) 0 from 1[1]
(18) 14 from 17
(19) 21 from 24
(20) 3 from 5[1]
(21) 32 from 38
(22) 14 from 15
(23) 45 from 48
(24) 14 from 18
(25) 12 from 16
(26) 24 from 29
(27) 42 from 48
(28) 81 from 89
(29) 32 from 36
(30) 42 from 47
(31) 0 from 6[1]
(32) 10 from 13
(33) 72 from 76
(34) 48 from 49
(35) 64 from 68
(36) 3 from 4[1]
(37) 81 from 84
(38) 54 from 57
(39) 10 from 12
(40) 24 from 26
(41) 72 from 79
(42) 36 from 38
(43) 0 from 2[1]
(44) 56 from 57
(45) 15 from 18
(46) 32 from 39
(47) 40 from 46
(48) 63 from 69
(49) 10 from 14
(50) 25 from 28
(51) 56 from 59
(52) 72 from 74
(53) 2 from 3[1]
(54) 81 from 88
(55) 35 from 39
(56) 40 from 44
(57) 15 from 19
(58) 63 from 66
(59) 20 from 24
(60) 54 from 58
(61) 72 from 73
(62) 0 from 4[1]
(63) 25 from 29
(64) 81 from 87
(65) 36 from 39
(66) 42 from 43
(67) 54 from 59
(68) 63 from 65
(69) 72 from 75
(70) 5 from 7[1]
(71) 25 from 27
(72) 81 from 86
(73) 32 from 37
(74) 56 from 58
(75) 42 from 46
(76) 64 from 67
(77) 0 from 3[1]
(78) 72 from 78
(79) 28 from 29
(80) 81 from 82
(81) 30 from 33
(82) 14 from 19
(83) 45 from 46
(84) 63 from 68
(85) 5 from 9[1]
(86) 72 from 77
(87) 27 from 29
(88) 81 from 83
(89) 35 from 36
(90) 16 from 18
(91) 42 from 44
(92) 63 from 67
(93) 0 from 8[1]
(94) 27 from 28
(95) 81 from 85
(96) 35 from 38
(97) 16 from 19
(98) 40 from 45
(99) 64 from 69

[1] The forty-five combinations of simple subtraction may be omitted if the pupils are well up in them.

(100) 4 from 5[1]
(101) 24 from 28
(102) 16 from 17
(103) 35 from 37
(104) 64 from 66
(105) 42 from 45
(106) 4 from 6[1]
(107) 21 from 27
(108) 15 from 17
(109) 36 from 37
(110) 64 from 65
(111) 40 from 43
(112) 0 from 5[1]
(113) 25 from 26
(114) 15 from 16
(115) 30 from 35
(116) 63 from 64
(117) 40 from 42
(118) 5 from 6[1]
(119) 21 from 26
(120) 12 from 14
(121) 32 from 35
(122) 6 from 8[1]
(123) 40 from 41
(124) 24 from 25
(125) 12 from 13
(126) 32 from 34
(127) 0 from 7[1]
(128) 21 from 25
(129) 10 from 11
(130) 30 from 34
(131) 4 from 7[1]
(132) 20 from 23
(133) 32 from 33
(134) 6 from 9[1]
(135) 21 from 22
(136) 6 from 7[1]
(137) 20 from 22
(138) 8 from 9[1]
(139) 14 from 16
(140) 0 from 9

THE MORE DIFFICULT GROUP

(141) 9 from 17[1]
(142) 72 from 80
(143) 49 from 55
(144) 18 from 26
(145) 63 from 70
(146) 28 from 33
(147) 56 from 63
(148) 9 from 10[1]
(149) 45 from 53
(150) 16 from 23
(151) 64 from 71
(152) 35 from 41
(153) 54 from 60
(154) 8 from 14[1]
(155) 49 from 54
(156) 18 from 21
(157) 27 from 31
(158) 36 from 43
(159) 56 from 61
(160) 9 from 15[1]
(161) 45 from 51
(162) 14 from 20
(163) 64 from 70
(164) 36 from 41
(165) 54 from 62
(166) 6 from 10[1]
(167) 48 from 55
(168) 18 from 25
(169) 36 from 42
(170) 28 from 31
(171) 9 from 13[1]
(172) 48 from 54
(173) 16 from 22
(174) 35 from 40
(175) 54 from 61
(176) 7 from 11[1]
(177) 49 from 53
(178) 18 from 20
(179) 36 from 40
(180) 56 from 60
(181) 8 from 12[1]
(182) 28 from 32
(183) 18 from 23
(184) 7 from 13[1]
(185) 48 from 50
(186) 16 from 21
(187) 9 from 16[1]
(188) 49 from 52
(189) 27 from 32
(190) 7 from 10[1]
(191) 45 from 52
(192) 16 from 20
(193) 8 from 15[1]
(194) 45 from 50
(195) 18 from 22
(196) 9 from 11[1]
(197) 48 from 51
(198) 27 from 33
(199) 8 from 10[1]
(200) 48 from 52
(201) 24 from 31
(202) 9 from 14[1]
(203) 49 from 50
(204) 24 from 30
(205) 6 from 11[1]
(206) 49 from 51

[1] The forty-five combinations of simple subtraction may be omitted if the pupils are well up in them.

(207) 27 from 35
(208) 8 from 13 [1]
(209) 28 from 30
(210) 8 from 11 [1]
(211) 27 from 34
(212) 9 from 12 [1]
(213) 28 from 34
(214) 7 from 12 [1]
(215) 27 from 30
(216) 36 from 44
(217) 63 from 71
(218) 56 from 62
(219) 48 from 53
(220) 18 from 24

THE THREE HUNDRED AND SIXTY COMBINATIONS IN SHORT DIVISION WHICH GIVE REMAINDERS

THE EASIER GROUP

1.	9)58 [2]	19.	6)14	37	9)28	55.	5)27
2.	2)3	20.	8)74	38	9)65	56	4)13
3	5)24	21.	5)18	39	4)37	57	8)35
4	8)59	22.	4)33	40	2)9	58	9)69
5	6)39	23	3)25	41	8)68	59	7)68
6	9)75	24	5)32	42.	4)23	60	9)37
7.	4)27	25	9)76	43	7)45	61	8)46
8	4)34	26	6)9	44	9)68	62.	3)7
9	6)15	27	4)15	45	4)17	63	5)49
10	8)42	28	7)48	46.	9)59	64	9)39
11	9)66	29	7)9	47	8)28	65	3)19
12	3)4	30.	6)17	48.	3)16	66.	6)45
13	5)22	31.	5)29	49	4)39	67	3)26
14	7)59	32.	2)11	50	9)77	68.	9)19
15	8)38	33	9)82	51	5)6	69	8)45
16.	7)29	34	8)69	52	7)66	70	8)66
17	3)5	35	6)26	53	8)47	71	5)7
18	5)26	36	5)34	54.	9)84	72	6)25

[1] The forty-five combinations of simple subtraction may be omitted if the pupils are well up in them

[2] Children are supposed to respond to 9)58 — six and four over. Similar responses are to be given to all these combinations.

73 6)58
74 7)22
75 8)79
76 7)27
77 3)13
78. 5)47
79 7)24
80 9)29
81 8)33
82 4)22
83 2)7
84 5)12
85 6)29
86 7)16
87 6)8
88 2)19
89 4)14
90 8)44
91 9)73
92 4)38
93 7)38
94 9)89
95 7)67
96. 4)19
97 6)55
98 7)36
99 8)43
100 5)16
101 6)19
102 7)17
103. 3)14
104 8)34
105 9)67
106 7)64
107 5)13
108 2)5
109. 6)7
110 9)64
111 9)55
112 8)9
113 7)8
114 2)13
115 7)26
116. 9)46
117 6)57
118 5)36
119 8)49
120 5)48
121 2)15
122 6)38
123 8)26
124 7)25
125 8)73
126 6)44
127 5)9
128 9)74
129 3)8
130 9)79
131 4)25
132 8)57
133 5)14
134 8)58
135 7)23
136 5)19
137 9)56
138 5)41
139 2)17
140 9)86
141 7)58
142 4)9
143 9)78
144 7)19
145 6)49
146 7)18
147 9)83
148 6)46
149 3)17
150 8)41
151 7)69
152 9)88
153. 7)37
154 6)35
155. 8)27
156 5)21
157 3)23
158 9)87
159. 8)19
160 5)23
161. 4)29
162 8)65
163 5)11
164 7)39
165 5)44
166 6)31
167 3)22
168 8)36
169 6)33
170 5)17
171 9)57
172 5)38
173 3)29
174 6)59
175 7)43
176 6)27
177 8)37
178 9)48
179 6)43
180 5)31
181 7)46
182 4)18
183. 5)33
184 6)37
185 3)28
186 6)16
187 8)77
188 6)13

189	5)42	198	7)15	207.	9)38	216	4)26
190	8)18	199	5)28	208	5)46	217	7)57
191	8)29	200	8)78	209	5)43	218	5)8
192	7)47	201	7)44	210	6)32	219	7)65
193.	8)75	202	9)49	211	4)21	220	6)56
194	5)37	203.	4)5	212	6)28	221.	4)6
195	4)7	204	8)67	213	9)85	222	8)17
196	9)47	205.	5)39	214	8)25	223	8)76
197.	6)47	206	4)35	215.	6)34	224	8)39

THE MORE DIFFICULT GROUP

225	9)32	244	8)14	263	8)60	282	7)55
226	8)15	245	6)51	264.	9)14	283.	9)2
227	6)53	246	9)3	265	9)20	284.	8)61
228	9)17	247	7)60	266	4)31	285	8)13
229	7)30	248	8)11	267	8)31	286	9)35
230	8)10	249	9)30	268	9)6	287	6)1
231	9)41	250.	7)6	269	9)71	288	6)50
232	7)13	251	6)23	270	9)26	289	9)11
233	9)34	252.	9)15	271	7)3	290	9)40
234	6)5	253	6)41	272	7)34	291	8)6
235	8)12	254	8)7	273.	9)13	292	9)25
236	9)12	255	9)60	274	8)55	293.	7)10
237	7)33	256.	4)10	275	6)2	294.	6)40
238	9)10	257	8)53	276	6)52	295	9)7
239	8)21	258.	9)16	277.	4)11	296	8)52
240	7)12	259	7)62	278	9)53	297	6)10
241	8)51	260	4)3	279	9)8	298	9)50
242	9)1	261	9)51	280	8)71	299	5)1
243	7)31	262	7)1	281	7)11	300	9)52

301	3)11	316	9)62	331.	4)30	346	6)21
302	9)43	317.	8)3	332	9)33	347.	4)2
303	5)3	318	7)51	333	8)1	348.	8)30
304.	7)40	319	9)22	334.	9)31	349.	7)2
305.	9)5	320	8)62	335.	3)2	350.	8)22
306	9)42	321.	6)4	336.	9)24	351.	2)1
307	6)11	322.	8)50	337	6)3	352	7)52
308	7)32	323	7)4	338	9)61	353.	9)21
309.	9)4	324	9)44	339.	8)2	354.	5)4
310.	6)22	325	3)20	340.	9)23	355	7)53
311	8)5	326	8)4	341	6)20	356	8)54
312.	7)54	327	5)2	342	9)80	357	7)50
313	3)1	328.	8)20	343	7)41	358.	8)70
314	7)61	329	4)1	344	8)23	359	7)20
315.	7)5	330.	9)70	345	8)63	360	3)10

EXERCISES INVOLVING SITUATIONS WHICH GIVE ZERO QUOTIENTS IN SHORT DIVISION

WITH OR WITHOUT REMAINDERS

(1)	9)54270	(11)	9)8138	(21)	8)5669	(31)	1)8709
(2)	8)8401	(12)	7)3546	(22)	2)1814	(32)	8)727
(3)	9)637	(13)	7)283	(23)	6)364	(33)	9)451
(4)	5)3504	(14)	7)5610	(24)	8)405	(34)	7)4202
(5)	9)7281	(15)	6)4836	(25)	5)4001	(35)	3)2701
(6)	6)4224	(16)	6)5405	(26)	8)4803	(36)	5)4538
(7)	7)6356	(17)	2)1609	(27)	5)202	(37)	4)3201
(8)	9)3654	(18)	4)3624	(28)	8)32240	(38)	8)66016
(9)	4)283	(19)	9)1804	(29)	9)2769	(39)	7)211407
(10)	6)3018	(20)	7)49630	(30)	3)324		

LONG DIVISION EXAMPLES GRADED ACCORDING TO DIFFICULTY

LIST A—SIMPLEST EXAMPLES

(1) 13)143
(2) 21)441
(3) 21)672
(4) 21)525
(5) 21)336
(6) 21)567
(7) 32)768
(8) 23)483
(9) 31)651
(10) 31)992
(11) 31)837
(12) 31)2387
(13) 17)187
(14) 31 2294
(15) 32)672
(16) 36)396
(17) 41)943
(18) 41)1394
(19) 41)656
(20) 41)2788
(21) 42)1386
(22) 51)561
(23) 51)4692
(24) 51)3774
(25) 51)2856
(26) 51)3978
(27) 52)1664
(28) 53)689
(29) 61)3782
(30) 61)2745
(31) 61)1769
(32) 62)1984
(33) 71)2982
(34) 71)5893
(35) 71)2769
(36) 72)1584
(37) 81)1863
(38) 81)3645

LIST B—EXAMPLES INVOLVING CARRYING AS THE ONLY NEW DIFFICULTY

(1) 13)182
(2) 23)1196
(3) 28)392
(4) 33)1782
(5) 34)2074
(6) 36)1476
(7) 42)3486
(8) 45)3150
(9) 47)1598
(10) 48)1488
(11) 52)2964
(12) 54)4482
(13) 56)3976
(14) 57)3477
(15) 58)2494
(16) 59)1888
(17) 62)4774
(18) 63)5796
(19) 65)5265
(20) 66)3564
(21) 67)4154
(22) 68)4080
(23) 74)6882
(24) 75)6225
(25) 76)5548
(26) 78)3978
(27) 86)7998
(28) 82)5986
(29) 84)4704
(30) 85)6970
(31) 87)5568
(32) 88)4664
(33) 93)6975
(34) 95)7885

LIST C—EXAMPLES INVOLVING BORROWING AS THE ONLY NEW DIFFICULTY

(1) $41\overline{)3444}$
(2) $42\overline{)1848}$
(3) $51\overline{)3621}$
(4) $61\overline{)2318}$
(5) $91\overline{)3913}$
(6) $91\overline{)5733}$
(7) $91\overline{)7553}$
(8) $92\overline{)1288}$
(9) $94\overline{)1974}$
(10) $61\overline{)5185}$
(11) $62\overline{)2728}$
(12) $71\overline{)2414}$
(13) $71\overline{)4828}$
(14) $71\overline{)6248}$
(15) $72\overline{)3168}$
(16) $81\overline{)1539}$
(17) $81\overline{)5427}$
(18) $81\overline{)7128}$
(19) $83\overline{)2573}$
(20) $91\overline{)1547}$
(21) $22\overline{)308}$
(22) $31\overline{)2046}$
(23) $33\overline{)1089}$
(24) $42\overline{)1008}$
(25) $51\overline{)4233}$
(26) $61\overline{)2501}$
(27) $62\overline{)2046}$
(28) $53\overline{)1219}$
(29) $73\overline{)2409}$
(30) $81\overline{)5508}$
(31) $91\overline{)2002}$
(32) $92\overline{)1104}$
(33) $93\overline{)3069}$

LIST D—EXAMPLES INVOLVING BOTH CARRYING AND BORROWING

(1) $43\overline{)3916}$
(2) $53\overline{)1961}$
(3) $54\overline{)4914}$
(4) $59\overline{)1357}$
(5) $63\overline{)2394}$
(6) $65\overline{)3445}$
(7) $69\overline{)1656}$
(8) $73\overline{)3358}$
(9) $74\overline{)6216}$
(10) $78\overline{)1248}$
(11) $82\overline{)3116}$
(12) $83\overline{)2241}$
(13) $84\overline{)3528}$
(14) $86\overline{)2752}$
(15) $87\overline{)7134}$
(16) $89\overline{)3115}$
(17) $93\overline{)4929}$
(18) $94\overline{)3290}$
(19) $95\overline{)3230}$
(20) $96\overline{)3552}$
(21) $97\overline{)4171}$
(22) $98\overline{)3528}$
(23) $99\overline{)4455}$
(24) $44\overline{)1100}$
(25) $54\overline{)3402}$
(26) $65\overline{)1040}$
(27) $72\overline{)4608}$
(28) $75\overline{)5400}$
(29) $82\overline{)2050}$
(30) $85\overline{)2125}$
(31) $88\overline{)2200}$
(32) $94\overline{)4042}$
(33) $96\overline{)4032}$
(34) $99\overline{)8019}$

LIST E—SIMPLEST EXAMPLES IN WHICH THE TRIAL QUOTIENT IS NOT THE TRUE QUOTIENT

(1) 12)600
(2) 12)1080
(3) 13)1040
(4) 14)840
(5) 15)450
(6) 15)1200
(7) 16)800
(8) 17)340
(9) 17)1190
(10) 18)720
(11) 18)1620
(12) 19)1140
(13) 23)1840
(14) 24)1920
(15) 25)1750
(16) 26)1560
(17) 27)1080
(18) 27)2430
(19) 28)1960
(20) 29)1160
(21) 34)2720
(22) 35)3150
(23) 36)3240
(24) 37)2960
(25) 38)3420
(26) 39)2730
(27) 45)4050
(28) 47)4230
(29) 48)3840
(30) 49)4410
(31) 58)4060
(32) 59)5310
(33) 69)5520
(34) 89)8010

LIST F—EXAMPLES OF MODERATE DIFFICULTY IN WHICH THE TRIAL QUOTIENT IS NOT THE TRUE QUOTIENT

(1) 45)3555
(2) 46)4462
(3) 47)4371
(4) 49)2793
(5) 52)3588
(6) 56)1624
(7) 57)3933
(8) 58)2842
(9) 59)2773
(10) 59)5369
(11) 64)5696
(12) 67)5762
(13) 68)6528
(14) 69)5589
(15) 73)4964
(16) 77)6468
(17) 78)6942
(18) 79)7347
(19) 88)5984
(20) 89)7298
(21) 46)2622
(22) 55)2640
(23) 56)5096
(24) 65)1820
(25) 67)4489
(26) 68)2448
(27) 75)3525
(28) 78)7332
(29) 86)6536
(30) 89)8633

LIST G—DIFFICULT EXAMPLES IN WHICH THE TRIAL QUOTIENT IS NOT THE TRUE QUOTIENT

(1) 47)4418	(12) 87)8004	(23) 52)5044
(2) 49)4116	(13) 98)4508	(24) 47)3008
(3) 56)4424	(14) 17)306	(25) 45)4005
(4) 59)3481	(15) 84)7308	(26) 42)4032
(5) 66)5214	(16) 23)2001	(27) 57)5073
(6) 74)7252	(17) 26)2158	(28) 56)1008
(7) 78)6162	(18) 58)4524	(29) 32)3008
(8) 86)1634	(19) 32)3008	(30) 58)4002
(9) 89)2492	(20) 37)3404	(31) 73)7008
(10) 96)8160	(21) 58)2030	(32) 87)8004
(11) 98)9310	(22) 45)4410	(33) 98)9016

LIST H—EASY PROBLEMS IN LONG DIVISION WITH MORE THAN TWO FIGURES IN THE QUOTIENT

(1) 13)2769	(11) 41)99999	(20) 61)45994
(2) 14)4961	(12) 32)6848	(21) 61)17995
(3) 22)6864	(13) 41)38991	(22) 41)129888
(4) 21)9996	(14) 51)29886	(23) 51)167994
(5) 21)19992	(15) 51)79764	(24) 81)27945
(6) 21)119994	(16) 41)25994	(25) 71)16969
(7) 31)7967	(17) 51)19992	(26) 81)439992
(8) 31)19995	(18) 61)149999	(27) 51)139485
(9) 31)139996	(19) 61)439993	(28) 51)108987
(10) 31)17794		

LIST I—EASY PROBLEMS IN LONG DIVISION WITH MORE THAN TWO FIGURES IN THE DIVISOR

(1) 111)8991
(2) 111)7992
(3) 111)469752
(4) 111)238983
(5) 112)2576
(6) 112)349888
(7) 113)1356
(8) 113)13899
(9) 114)2394
(10) 123)14883
(11) 122)14762
(12) 122)147986
(13) 121)158994
(14) 122)147864
(15) 121)148951
(16) 121)499972
(17) 131)2751
(18) 121)498883
(19) 3124)37488
(20) 1122)135662

LIST J—EXERCISES IN LONG DIVISION WITH ZEROS IN THE QUOTIENT

(1) 67)2707
(2) 96)19207
(3) 58)53606
(4) 74)518370
(5) 175)685397
(6) 286)87516
(7) 391)1566737
(8) 458)430973
(9) 567)453749
(10) 645)4553700
(11) 794)638749
(12) 856)496729
(13) 932)804757
(14) 719)432818
(15) 841)235845
(16) 932)4667456
(17) 493)152610
(18) 372)2622775
(19) 627)769676
(20) 961)6775050
(21) 573)345708
(22) 481)3126896
(23) 168)1517040
(24) 298)2086894
(25) 385)1570909
(26) 192)153459

FORM 8 Do Not Teach What the Child Knows.
Teach What He Does Not Know

WISCONSIN INVENTORY TESTS

ARITHMETIC—LONG DIVISION

Devised by WORTH J OSBURN

Directions to be read by teacher and pupils together

The exercises on this page are arranged in the form of a ladder. Sometimes you think you have gotten to the top when you have not Wrong answers do not count See if you can climb the ladder The examples get more difficult as you go up Begin *at the bottom* and *work upward* Watch your step and don't get dizzy. Put all of your work on this sheet Do not check your work, but when you have finished, ask your teacher to read the correct answers Put crosses in the ladder to show how high you have climbed before you made a mistake. You can climb no higher than your first mistake.

(10) 6597)5769000 and over.

(9) 27)17523 and over.

(8) 78)2262 and over.

(7) 28)958 and over

(6) 68)5448 and over.

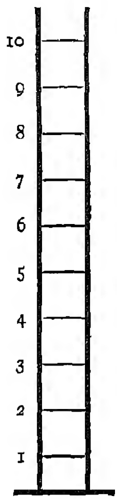
(5) 97)7087 and over

(4) 32)1312 and over

(3) 85)5365 and over

(2) 43)989 and over.

(1) 90)5454 and over.



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